Determining travellings about $\mathcal{P}^2$, $\mathcal{P}$ and the identity functor on $\underline{\mathbf{Ens}}$

Pierre Damphousse[1]   René Guitart[2]

NatTrans2.tex → 30 Juillet 2001, 13:50 heures

[1]Département de mathématiques, Faculté des sciences et techniques, Parc de Grandmont, Tours 37200 ; damphous@univ-tours.fr

[2]Université Paris 7, URA 212 du CNRS, Case postale 7012, 2 place Jussieu, Paris 75251 Cedex 05 ; guitart@mathp7.jussieu.fr

Contents

1 The task : Nature and Tools **4**

1.0 Revisiting some usual notations . . . 4

1.1 Functorialities of the power set construction $\mathcal{P}$ . . . 5

1.2 Nature of the task . . . 6

1.3 The grammar of companion squares . . . 8

1.4 Adjointness and ordered sets (AOS) . . . 14

2 Traveling about the identity and $\mathcal{P}$ **18**

2.1 Between $\mathbf{Id}_{\mathbf{Ens}}$ and $\mathcal{P}$ . . . 18

2.2 About $\mathcal{P}$ . . . 20

2.2.1 The covariant→covariant case . . . 20

2.2.2 The contravariant→contravariant case . . . 24

2.2.3 Linking results with the principle of adjoint naturality . . . 24

3 Traveling from $\mathcal{P}$ to $\mathcal{P}^2$ **26**

3.1 The covariant→covariant case . . . 26

3.1.1 Case $\exists \to \exists\,\exists$ . . . 26

3.1.2 Case $\exists \to \exists\,\forall$ . . . 35

3.1.3 Case $\exists \to \forall\,\exists$ . . . 35

3.1.4 Case $\exists \to \forall\,\forall$ . . . 35

3.1.5 Case $\exists \to C\,C$ . . . 35

3.1.6 Case $\forall \to \exists\,\exists$ . . . 35

3.1.7 Case $\forall \to \exists\,\forall$ . . . 35

3.1.8 Case $\forall \to \forall\,\exists$ . . . 35

3.1.9 Case $\forall \to \forall\,\forall$ . . . 35

3.1.10 Case $\forall \to C\,C$ . . . 35

3.2 The contravariant→contravariant case . . . 35

3.2.1 The π-ψ-cube . . . 35

3.2.2 The logical neighbourhood of the π-ψ-cube . . . 37

3.2.3 Enumerating the transformations . . . 38

“contentsline subsubsection3.2.4SCdd41 “contentsline subsubsection3.2.5SCde44 “contentsline subsubsection3.2.6SCdf44 “contentsline subsubsection3.2.7SCdg50

4 Traveling from $\mathcal{P}^2$ to $\mathcal{P}$ **54**

5 Traveling from $\mathcal{P}^2$ to $\mathcal{P}^2$ **54**

Abstract

NT001.txt The Cantor's *power set* construction of the set $\mathcal{P}X$ of all subsets of X may be considered as a functor P_1 in three ways, one contravariant and two covariant; and then $\mathcal{P}^2X = \mathcal{P}\mathcal{P}X$ is a functor P_2, through composition of the various P_1's, in nine ways (four contravariants and five covariants). We give a complete analysis of all functorial (*i.e.* monotonous) transformations between these functors.

1 The task : Nature and Tools

1.0 Revisiting some usual notations

NT002a.txt Usual notations regarding power sets and images of functions entails an ambiguity which is generally unnoticeable, but which is unmanageable in this work. For this reason, we make the following conventions.

If $f : X \to Y$ is a mapping and x an element of X, mathematicians usually write $f(x)$ for the element of Y associated to x by f —the image of x by f; we shall always write fx for this element of Y.

If A is a subset of X, mathematicians usually write $f(A)$ the subset of Y made of all $f(x)$'s when x runs through A —the direct image of A. We will keep this notation, but we shall often replace it by $\underline{\exists} fA$, because of technical requirements that we will describe on due time.

Of course, the empty set is always written $\emptyset$. If $x \in X$, we write $\{x\}$ the "singteton" associated with x, *i.e.* the subset of X with x as its unique element.

With the usual notation, we always have $\{f(a)\} = f(\{a\})$. With our notation, we have $\{fa\} = f(\{a\}) = \underline{\exists} f\{a\}$; we always have $\emptyset = f(\emptyset) = \underline{\exists} f\emptyset$, but we may also have $f\emptyset \neq \emptyset$. More generally, if X is an ordinal ν and if $\mu < \nu$, μ is both an element and a subset of X; the usual notation $f(\nu)$ has then two different meanings, which are, in our notation, $f\nu$ and $f(\nu)$ (or $\underline{\exists} f\nu$).

If X is a set, we write $\mathcal{P}X$ the set of all its subsets, $\mathcal{P}^2X$ the set of all subsets of $\mathcal{P}X$, *etc.* We make the convention to use different fonts, when possible, to denote elements of X, $\mathcal{P}X$, $\mathcal{P}^2X$, ... *etc.* We shall write:

0) a, b, ... (lower case) the elements of X.

1) A, B, ... (upper case) the elements of $\mathcal{P}X$.

2) $\mathbb{A}$, $\mathbb{B}$, ... the elements of $\mathcal{P}^2X$.

3) $\mathcal{A}$, $\mathcal{B}$, ... the elements of $\mathcal{P}^3X$.

In the context of the technical requirements already mentionnned and at the center of this work, $\mathcal{P}X$ will often be written $\underline{\mathbf{C}}X$, $\underline{\exists}X$, $\underline{\forall}X$ or with some variants of these (like $\underset{C}{\underline{\exists}}X$) depending on the situation.

Thus (exercise) any element B of $\underline{\exists}\,\underline{\exists} f\mathbb{A}$ is a subset of Y which is the direct image $f(A)$ by f of an element A of $\mathbb{A}$.

These conventions are sufficient for our work, but a systematic study of the $\mathcal{P}^n$'s would require a more general system of notation.

This notation is not pedantic; it allows for a “blind-folded calculation” when going up and down various levels of iterations of the power set construction. Moreover, it allows a simple and precise description (*v.g.* $\exists\exists f\mathbb{A}$) of sets which may be described in words, but with fancy steps and rounds.

1.1 Functorialities of the power set construction $\mathcal{P}$

NT002b.txt

For any set X, let $\mathcal{P}_{\subset} X = (\mathcal{P}X, \subset)$ be $\mathcal{P}X$ ordered with the inclusion, and $\mathcal{P}_{\supset} X = (\mathcal{P}X, \supset)$ be $\mathcal{P}X$ ordered with the inverse inclusion (“contains”). Let $\underline{\mathbf{Ens}}$ be the category of sets and mappings, $\underline{\mathbf{Ord}}$ be the category of ordered sets and “increasing” mappings. We define two functors $\underline{\mathbf{C}}_{\subset}$ and $\underline{\mathbf{C}}_{\supset} : \underline{\mathbf{Ens}}^{\mathrm{op}} \to \underline{\mathbf{Ord}}$ through:

- for a set X, $\underline{\mathbf{C}}_{\subset} X = \mathcal{P}_{\subset} X$ and $\underline{\mathbf{C}}_{\supset} X = \mathcal{P}_{\supset} X$;
- for a mapping $f : X \to Y$, $\underline{\mathbf{C}}_{\subset} f = f^* = \underline{\mathbf{C}}_{\supset} f$, where $f^* : \mathcal{P}Y \to \mathcal{P}X$ is defined as $f^*B = \{x \in X;\ \ fx \in B\}$ (the inverse image of B).

When $\underline{\mathbf{C}}_{\subset}$ and $\underline{\mathbf{C}}_{\supset}$ are followed by the forgetful functor $\underline{\mathbf{U}} : \underline{\mathbf{Ord}} \to \underline{\mathbf{Ens}}$, we obtain a single functor $\underline{\mathbf{C}} : \underline{\mathbf{Ens}}^{\mathrm{op}} \to \underline{\mathbf{Ens}}$. We say that $\underline{\mathbf{C}}$ has two *spins*, written $\subset$ and $\supset$, to mean that it may be seen as $\underline{\mathbf{C}} = \underline{\mathbf{U}}\,\underline{\mathbf{C}}_{\subset}$ and $\underline{\mathbf{C}} = \underline{\mathbf{U}}\,\underline{\mathbf{C}}_{\supset}$ respectively.

From the fundamental conception of quantifiers due to Lawvere, for each $f : X \to Y$, the functor $\underline{\mathbf{C}}_{\subset} f : \underline{\mathbf{C}}_{\subset} Y \to \underline{\mathbf{C}}_{\subset} X$ has a left-adjoint $\exists_{\subset} f$ and a right-adjoint $\forall_{\subset} f$ (*see* [1]), a fact written

$$\exists_{\subset} f \dashv \underline{\mathbf{C}}_{\subset} f \dashv \forall_{\subset} f \qquad \text{and} \qquad \forall_{\supset} f \dashv \underline{\mathbf{C}}_{\supset} f \dashv \exists_{\supset} f \qquad\qquad \text{(quantification)}$$

namely the functors given by

$$\begin{aligned} \exists_{\subset} f : (\mathcal{P}X, \subseteq) \to (\mathcal{P}Y, \subseteq) \quad &: \quad A \mapsto \exists_{\subset} f A = \big\{ y \in Y;\ \exists x \big((y = fx) \wedge (x \in A) \big) \big\} \\ \forall_{\subset} f : (\mathcal{P}X, \subseteq) \to (\mathcal{P}Y, \subseteq) \quad &: \quad A \mapsto \forall_{\subset} f A = \big\{ y \in Y;\ \forall x \big((y = fx) \Rightarrow (x \in A) \big) \big\}. \end{aligned}$$

Of course, $\exists_{\subset} f A = \exists_{\supset} f A$ is just the direct image of A by f (*i.e.* $f(A)$). We shall use these notations when we systematically consider the direct image construction as a functor.

The four constructions $\underline{\mathbf{Ens}} \to \underline{\mathbf{Ord}}$ given by

$$(1)\ :\ \begin{array}{c} X \\ \big\downarrow f \\ Y \end{array} \longmapsto \begin{array}{c} \mathcal{P}_{\subset} X \\ \big\downarrow \exists_{\subset} f \\ \mathcal{P}_{\subset} Y \end{array} \qquad\qquad (2)\ :\ \begin{array}{c} X \\ \big\downarrow f \\ Y \end{array} \longmapsto \begin{array}{c} \mathcal{P}_{\supset} X \\ \big\downarrow \exists_{\supset} f \\ \mathcal{P}_{\supset} Y \end{array}$$

$$(3)\ :\ \begin{array}{c} X \\ \Big\downarrow f \\ Y \end{array} \longmapsto \begin{array}{c} \mathcal{P}_{\subset} X \\ \Big\downarrow \underline{\forall}_{\subset} f \\ \mathcal{P}_{\subset} Y \end{array} \qquad\qquad (4)\ :\ \begin{array}{c} X \\ \Big\downarrow f \\ Y \end{array} \longmapsto \begin{array}{c} \mathcal{P}_{\supset} X \\ \Big\downarrow \underline{\forall}_{\supset} f \\ \mathcal{P}_{\supset} Y \end{array}$$

define four *covariant* functors $\underline{\mathbf{Ens}} \to \underline{\mathbf{Ord}}$; we write them $\underline{\exists}_{\subset}$, $\underline{\exists}_{\supset}$, $\underline{\forall}_{\subset}$ and $\underline{\forall}_{\supset}$. When $\underline{\exists}_{\subset}$ and $\underline{\exists}_{\supset}$ (resp. $\underline{\forall}_{\subset}$ and $\underline{\forall}_{\supset}$) are followed by $\underline{\mathbf{U}}$, we obtain a single functor $\underline{\exists} : \underline{\mathbf{Ens}} \to \underline{\mathbf{Ens}}$ (resp. $\underline{\forall} : \underline{\mathbf{Ens}} \to \underline{\mathbf{Ens}}$). As for $\underline{\mathbf{C}}$, we say that $\underline{\exists}$ and $\underline{\forall}$ have two spins, written $\subset$ and $\supset$ to mean that they may be seen as $\underline{\exists} = \underline{\mathbf{U}}\,\underline{\exists}_{\subset}$ or $\underline{\exists} = \underline{\mathbf{U}}\,\underline{\exists}_{\supset}$, and $\underline{\forall} = \underline{\mathbf{U}}\,\underline{\forall}_{\subset}$ or $\underline{\forall} = \underline{\mathbf{U}}\,\underline{\forall}_{\supset}$ respectively.

Of course, $\underline{\exists}X$, $\underline{\exists}_{\subset}X$, $\underline{\exists}_{\supset}X$, $\underline{\mathbf{C}}X$, *etc.* are just (as sets) $\mathcal{P}X$. We shall use these notations when the power set construction is considered as functorial in some appropriate sense.

We write $\nu_X : \mathcal{P}X \to \mathcal{P}X$ the mapping defined through $A \mapsto \complement_X A$ (the complement of A in X, which we shall now write $X \backslash A$). Given $f : X \to Y$, we have $\underline{\forall}_{\subset} f = \nu_Y\, \underline{\exists}_{\supset} f\, \nu_X$, $\underline{\mathbf{C}}_{\subset} f = \nu_Y\, \underline{\mathbf{C}}_{\supset} f\, \nu_X$, *etc.* The ν_X are the components of various natural transformations that we all write ν, as shown in Figure 1.

1.2 Nature of the task

NT002c.txt

Let us now work out functors, both co- and contravariant from $\underline{\mathbf{Ens}}$ to $\underline{\mathbf{Ens}}$, the values of which on a set X are $\mathcal{P}X$ and $\mathcal{P}^2X$, using the functorial contents of $\mathcal{P}$ given above. There are altogether 12 possibilities, as given in the following table:

	Value on a set X	Covariant	Contravariant
1	$\mathcal{P}X$	$\underline{\exists}\quad \underline{\forall}$	$\underline{\mathbf{C}}$
2	$\mathcal{P}^2X$	$\underline{\mathbf{C}}^2$ $\underline{\exists}^2\quad \underline{\exists}\,\underline{\forall}$ $\underline{\forall}^2\quad \underline{\forall}\,\underline{\exists}$	$\underline{\mathbf{C}}\,\underline{\exists}\quad \underline{\exists}\,\underline{\mathbf{C}}$ $\underline{\mathbf{C}}\,\underline{\forall}\quad \underline{\forall}\,\underline{\mathbf{C}}$

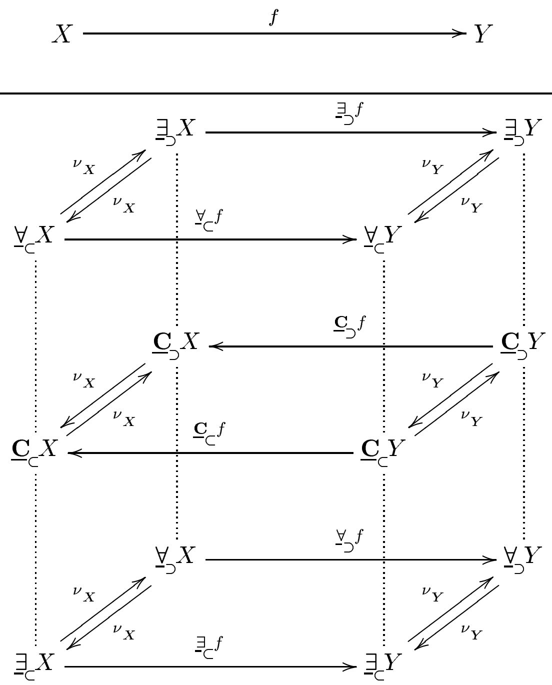
Let us use $\mathcal{P}$ to mean any of the three functors on line 1, and $\mathcal{P}^2$ to mean any of the nine functors on line 2. Our goal is to describe all natural transformations in each of the following cases:

$$\mathcal{P} \to \mathcal{P} \qquad \mathcal{P} \to \mathcal{P}^2 \qquad \mathcal{P}^2 \to \mathcal{P} \qquad \mathcal{P}^2 \to \mathcal{P}^2$$

Each of these cases subdivides in turn in two subcases:

1: The last two cases have not yet been investigated

(a) Contravariant $\to$ Contravariant (b) Covariant $\to$ Covariant



Arrows at the boundary of each "horizontal rectangle" are functors; each of them is left-adjoint to its above neighbour. The various ν_X, ν_Y, ..., *etc.* are the components of natural transformations $\underline{\forall}_{\subset} \to \underline{\exists}_{\supset}$, $\underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\supset}$, ..., *etc.*

Figure 1: Functorialities of the Power Set construction $\mathcal{P}$.

This subdivision into cases and subcases correspond to sections and subsections below.

Among the natural transformations to be described, some have components which are functors (between ordered sets) for some choice of spins; we call this *componentwise functoriality* (cw-functoriality). Moreover, among these ones, some have components which all possess an adjoint (left or right depending on the choice of the spins) and among these last ones, some have components which all determine their domain as algebraic over their codomain. For those two cases, we speak respectively of componentwise adjointness and of componentwise algebraicity (cw-adjointness and cw-algebraicity).

Prior to any investigation, because $\underline{\mathbf{C}}$ is representable, there is an elementary argument, concerning transformations from $\underline{\mathbf{C}}$ to itself or to a contravariant functor of line 2, coming from the Yoneda lemma.

Proposition 1 *a) There are 4 natural transformations from* $\underline{\mathbf{C}}$ *to itself. b) If* $\underline{\mathbf{F}}$ *is any one of the four contravariant functors* $\underline{\mathbf{C}}\,\underline{\exists}$, $\underline{\mathbf{C}}\,\underline{\forall}$, $\underline{\exists}\,\underline{\mathbf{C}}$, *and* $\underline{\forall}\,\underline{\mathbf{C}}$, *there are 16 natural transformations from* $\underline{\mathbf{C}}$ *to* $\underline{\mathbf{F}}$.

Indeed, $\underline{\mathbf{C}}$ is representable: $\underline{\mathbf{C}}X = \underline{\mathbf{Ens}}(X, \mathbf{2})$, with $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$; from the Yoneda lemma, we have

$$\mathrm{Nat}(\underline{\mathbf{C}}, \underline{\mathbf{F}}) \cong \mathrm{Nat}(\underline{\mathbf{Ens}}(-, \mathbf{2}), \underline{\mathbf{F}}) \cong \underline{\mathbf{F}}\mathbf{2};$$

when on the one hand, $\underline{\mathbf{F}}$ is $\underline{\mathbf{C}}$, $\underline{\mathbf{F}}\mathbf{2}$ contains 4 elements, whence (a), and when on the other hand $\underline{\mathbf{F}}$ is one of the four given contravariant functors, $\underline{\mathbf{F}}\mathbf{2}$ contains 16 elements, whence (b).

Another interesting consequence of Yoneda is that the set $\mathrm{Nat}(\underline{\mathbf{C}}, \underline{\mathbf{F}})$, for any contravariant endofunctor $\underline{\mathbf{F}}$ of $\underline{\mathbf{Ens}}$, naturally inherits any structure of $\underline{\mathbf{F}}\mathbf{2}$ (here a boolean algebra structure).

2: A)With G any non representable functor in Ens, has Nat(G,F) a natural boolean structure? B) What is the real of structure de P2 ? A deductive set *à la* (Lambek-Scott)?

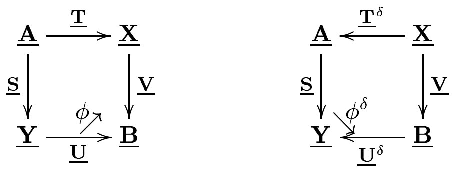
1.3 The grammar of companion squares

NT002d.txt

We suppose in this section that, for each functor $\underline{\mathbf{U}}$ with a right adjoint, such an adjoint is chosen, which we write $\underline{\mathbf{U}}^{\delta}$, and that, for each functor $\underline{\mathbf{V}}$ with a left-adjoint, such an adjoint is chosen, which we write $\underline{\mathbf{V}}^{\sigma}$, in such a way that $\underline{\mathbf{U}}^{\delta\sigma} = \underline{\mathbf{U}}$ and $\underline{\mathbf{V}}^{\sigma\delta} = \underline{\mathbf{V}}$.

Let us be given a square of functors like square (a) below together with a natural transformation $\phi : \underline{\mathbf{U}}\,\underline{\mathbf{S}} \to \underline{\mathbf{V}}\,\underline{\mathbf{T}}$. Moreover, let $\underline{\mathbf{T}}$ and $\underline{\mathbf{U}}$ have right-adjoints, say $\underline{\mathbf{T}}^{\delta}$ and $\underline{\mathbf{U}}^{\delta}$ respectively. Then, there is a natural transformation $\phi^{\delta} : \underline{\mathbf{S}}\,\underline{\mathbf{T}}^{\delta} \to \underline{\mathbf{U}}^{\delta}\,\underline{\mathbf{V}}$, that is square (a^{δ}) below is commutative up to within a natural transformation.

(δ)



(a) (a^δ)

In this case, when adjoints exist, ϕ^δ is an isomorphism if and only if square (a) is exact in the sense of GUITART (*see* [2]). To compute ϕ^δ, we first compose ϕ with $\underline{\mathbf{U}}^\delta$ and $\underline{\mathbf{T}}^\delta$:

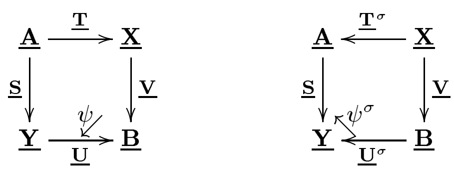
$$\underline{\mathbf{U}}^\delta \phi \underline{\mathbf{T}}^\delta \;:\; \underline{\mathbf{U}}^\delta\, \underline{\mathbf{U}}\, \underline{\mathbf{S}}\, \underline{\mathbf{T}}^\delta \longrightarrow \underline{\mathbf{U}}^\delta\, \underline{\mathbf{V}}\, \underline{\mathbf{T}}\, \underline{\mathbf{T}}^\delta.$$

Writing η for the unit of the adjunction $\underline{\mathbf{U}} \dashv \underline{\mathbf{U}}^\delta$ and ϵ for the counit of the adjunction $\underline{\mathbf{T}} \dashv \underline{\mathbf{T}}^\delta$, we obtain $\phi^\delta : \underline{\mathbf{S}}\,\underline{\mathbf{T}}^\delta \to \underline{\mathbf{U}}^\delta\,\underline{\mathbf{V}}$ as the composite

$$\underline{\mathbf{S}}\,\underline{\mathbf{T}}^\delta \xrightarrow{\eta \underline{\mathbf{S}}\,\underline{\mathbf{T}}^\delta} \underline{\mathbf{U}}^\delta\,\underline{\mathbf{U}}\,\underline{\mathbf{S}}\,\underline{\mathbf{T}}^\delta \xrightarrow{\underline{\mathbf{U}}^\delta \phi \underline{\mathbf{T}}^\delta} \underline{\mathbf{U}}^\delta\,\underline{\mathbf{V}}\,\underline{\mathbf{T}}\,\underline{\mathbf{T}}^\delta \xrightarrow{\underline{\mathbf{U}}^\delta\,\underline{\mathbf{V}}\epsilon} \underline{\mathbf{U}}^\delta\,\underline{\mathbf{V}}$$

The rule applied to go from square (a) to square (a^δ) will be called *the δ-rule*. In a parallel manner, if $\underline{\mathbf{T}}$ and $\underline{\mathbf{U}}$ have left-adjoints $\underline{\mathbf{T}}^\sigma$ and $\underline{\mathbf{U}}^\sigma$ respectively, and if we have a natural transformation $\psi : \underline{\mathbf{V}}\,\underline{\mathbf{T}} \to \underline{\mathbf{U}}\,\underline{\mathbf{S}}$, we have *the σ-rule*,

(σ)



(b) (b^σ)

with $\psi^\sigma : \underline{\mathbf{U}}^\sigma\,\underline{\mathbf{V}} \to \underline{\mathbf{S}}\,\underline{\mathbf{T}}^\sigma$, given by

$$\underline{\mathbf{U}}^\sigma\,\underline{\mathbf{V}} \xrightarrow{\underline{\mathbf{U}}^\sigma\,\underline{\mathbf{V}}\eta} \underline{\mathbf{U}}^\sigma\,\underline{\mathbf{V}}\,\underline{\mathbf{T}}\,\underline{\mathbf{T}}^\sigma \xrightarrow{\underline{\mathbf{U}}^\sigma \psi \underline{\mathbf{T}}^\sigma} \underline{\mathbf{U}}^\sigma\,\underline{\mathbf{U}}\,\underline{\mathbf{S}}\,\underline{\mathbf{T}}^\sigma \xrightarrow{\epsilon \underline{\mathbf{S}}\,\underline{\mathbf{T}}^\sigma} \underline{\mathbf{S}}\,\underline{\mathbf{T}}^\sigma$$

where η is the unit for $\underline{\mathbf{T}}^\sigma \dashv \underline{\mathbf{T}}$ and ϵ is the co-unit for$\underline{\mathbf{U}}^\sigma \dashv \underline{\mathbf{U}}$. Squares (a) and (a^δ) [resp. (b) and (b^σ)] are said to be *companion squares*.

One easily sees that (1) no rule can be established from squares (x) and (y):

$$
\begin{array}{ccc}
\underline{\mathbf{A}} & \xrightarrow{\underline{\mathbf{T}}} & \underline{\mathbf{X}} \\
{\scriptstyle \underline{\mathbf{S}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \underline{\mathbf{V}}} \\
\underline{\mathbf{Y}} & \xrightarrow[\underline{\mathbf{U}}]{\swarrow} & \underline{\mathbf{B}}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\underline{\mathbf{A}} & \xrightarrow{\underline{\mathbf{T}}} & \underline{\mathbf{X}} \\
{\scriptstyle \underline{\mathbf{S}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \underline{\mathbf{V}}} \\
\underline{\mathbf{Y}} & \xrightarrow[\underline{\mathbf{U}}]{\nearrow} & \underline{\mathbf{B}}
\end{array}
$$

$$
(\underline{\mathbf{T}} \dashv \underline{\mathbf{T}}^\delta,\ \underline{\mathbf{U}} \dashv \underline{\mathbf{U}}^\delta) \qquad\qquad (\underline{\mathbf{T}}^\sigma \dashv \underline{\mathbf{T}},\ \underline{\mathbf{U}}^\sigma \dashv \underline{\mathbf{U}})
$$

(x) (y)

that (2) if a square is commutative up to within a natural transformation and if the "vertical" functors have right-adjoints (resp. left-adjoints), then we have essentially squares of the form (a), (b), (x) and (y) as seen when oparating a diagonal symetry fixing A and B; therefore, (a), (b), (x) and (y) are essentially the only configurations; that (3) given ϕ as in case (σ), we have $((\phi^{\mathrm{op}})^\delta)^{\mathrm{op}} = \phi^\sigma$, or given ψ as in case (δ), $((\psi^{\mathrm{op}})^\sigma)^{\mathrm{op}} = \psi^\delta$, and that, when both left- and right-adjoints exist, $(\phi^\delta)^\sigma = \phi$ and $(\psi^\sigma)^\delta = \psi$. Hence, the δ-rule and the σ-rule are dual conjugate and inverse of each other, *i.e.* the diagram of squares in Figure 2 is commutative. Finally, when ϕ is the identity transformation $\underline{\mathbf{Id}}\,\underline{\mathbf{F}} \to \underline{\mathbf{Id}}\,\underline{\mathbf{F}}$, that is $\begin{array}{ccc}\cdot & \xrightarrow{F} & \cdot \\ {\scriptstyle \mathbf{1}}\downarrow & \overset{=}{\nearrow} & \downarrow{\scriptstyle \mathbf{1}} \\ \cdot & \xrightarrow[F]{} & \cdot\end{array}$ and $\begin{array}{ccc}\cdot & \xrightarrow{F} & \cdot \\ {\scriptstyle \mathbf{1}}\downarrow & \overset{=}{\swarrow} & \downarrow{\scriptstyle \mathbf{1}} \\ \cdot & \xrightarrow[F]{} & \cdot\end{array}$ respectively, then ϕ^δ and ϕ^σ are the identity transformations on $\underline{\mathbf{F}}^\delta$ and $\underline{\mathbf{F}}^\sigma$ respectively.

The case when right-adjoints (resp. left-adjoints) also have right-adjoints (resp. left-adjoints) yields a chain of relations when the δ-rule (resp. σ-rule) is iterated. Here is an account of the "language" of these relations.

Let us first consider the case of right-adjoints. When starting for example from $\underline{\mathbf{U}}\,\underline{\mathbf{S}} \longrightarrow \underline{\mathbf{V}}\,\underline{\mathbf{T}}$, we have the following sequence of natural transformations (read downwards):

$$
(\Delta) \qquad
\left.
\begin{array}{rrcl}
0) & \underline{\mathbf{U}}\,\underline{\mathbf{S}} & \longrightarrow & \underline{\mathbf{V}}\,\underline{\mathbf{T}} \\
-1) & \underline{\mathbf{U}}^\delta\,\underline{\mathbf{V}} & \longleftarrow & \underline{\mathbf{S}}\,\underline{\mathbf{T}}^\delta \\
-2) & \underline{\mathbf{T}}^\delta\,\underline{\mathbf{V}}^\delta & \longrightarrow & \underline{\mathbf{S}}^\delta\,\underline{\mathbf{U}}^\delta \\
-3) & \underline{\mathbf{T}}^{\delta\delta}\,\underline{\mathbf{S}}^\delta & \longleftarrow & \underline{\mathbf{U}}^{\delta\delta}\,\underline{\mathbf{V}}^\delta \\
-4) & \underline{\mathbf{U}}^{\delta\delta}\,\underline{\mathbf{S}}^{\delta\delta} & \longrightarrow & \underline{\mathbf{V}}^{\delta\delta}\,\underline{\mathbf{T}}^{\delta\delta} \\
-5) & \underline{\mathbf{U}}^{\delta\delta\delta}\,\underline{\mathbf{V}}^{\delta\delta} & \longleftarrow & \underline{\mathbf{S}}^{\delta\delta}\,\underline{\mathbf{T}}^{\delta\delta\delta} \\
\vdots & & \vdots &
\end{array}
\right\}
\ \Big\downarrow\ \delta\text{-rule}
$$

One may compute directly how to go from one line to the next through writing the squares and working out the result. One may also proceed "syntaxically"

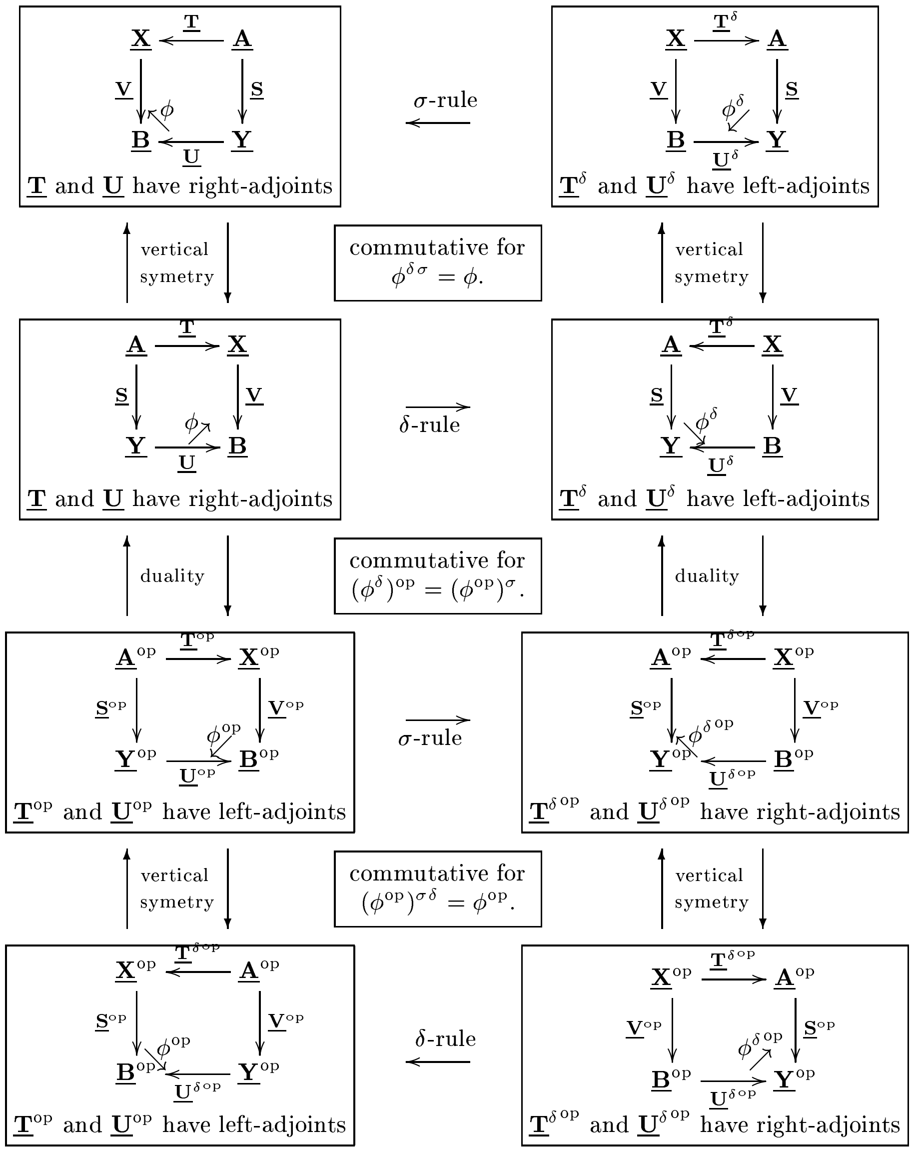


Figure 2: Companion squares

using the following grammar:

δ-1) Write the given natural transformation with a <u>right</u> arrow: .... $\underline{\mathbf{U}}\,\underline{\mathbf{S}} \longrightarrow \underline{\mathbf{V}}\,\underline{\mathbf{T}}$

δ-2) Consider this writing as a five-letter word, obtained as the concatenation of three words as follows: .... $\boxed{\underline{\mathbf{U}}}\,\boxed{\underline{\mathbf{S}} \longrightarrow \underline{\mathbf{V}}}\,\boxed{\underline{\mathbf{T}}}$

δ-3) Reverse the middle "diagram": ..... $\boxed{\underline{\mathbf{U}}}\,\boxed{\underline{\mathbf{V}} \longleftarrow \underline{\mathbf{S}}}\,\boxed{\underline{\mathbf{T}}}$

$\boxed{\underline{\mathbf{U}}^{\delta}}\,\boxed{\underline{\mathbf{V}} \longleftarrow \underline{\mathbf{S}}}\,\boxed{\underline{\mathbf{T}}^{\delta}}$

δ-5) Forget the formal factorisation in three words, and read the result as a diagram: .... $\underline{\mathbf{U}}^{\delta}\,\underline{\mathbf{V}} \longleftarrow \underline{\mathbf{S}}\,\underline{\mathbf{T}}^{\delta}$

Let us next consider the case of left-adjoints. If we start for example from $\underline{\mathbf{V}}\,\underline{\mathbf{T}} \longleftarrow \underline{\mathbf{U}}\,\underline{\mathbf{S}}$, we have the following sequence of natural transformations (read upwards):

$$(\Sigma) \qquad \sigma\text{-rule}\ \Big\uparrow \left\{ \begin{array}{rrcl} & \vdots & & \vdots \\ +5) & \underline{\mathbf{V}}^{\sigma\sigma\sigma}\,\underline{\mathbf{U}}^{\sigma\sigma} & \longrightarrow & \underline{\mathbf{T}}^{\sigma\sigma}\,\underline{\mathbf{S}}^{\sigma\sigma\sigma} \\ +4) & \underline{\mathbf{V}}^{\sigma\sigma}\,\underline{\mathbf{T}}^{\sigma\sigma} & \longleftarrow & \underline{\mathbf{U}}^{\sigma\sigma}\,\underline{\mathbf{S}}^{\sigma\sigma} \\ +3) & \underline{\mathbf{S}}^{\sigma\sigma}\,\underline{\mathbf{T}}^{\sigma} & \longrightarrow & \underline{\mathbf{U}}^{\sigma}\,\underline{\mathbf{V}}^{\sigma\sigma} \\ +2) & \underline{\mathbf{S}}^{\sigma}\,\underline{\mathbf{U}}^{\sigma} & \longleftarrow & \underline{\mathbf{T}}^{\sigma}\,\underline{\mathbf{V}}^{\sigma} \\ +1) & \underline{\mathbf{V}}^{\sigma}\,\underline{\mathbf{U}} & \longrightarrow & \underline{\mathbf{T}}\,\underline{\mathbf{S}}^{\sigma} \\ 0) & \underline{\mathbf{V}}\,\underline{\mathbf{T}} & \longleftarrow & \underline{\mathbf{U}}\,\underline{\mathbf{S}} \end{array} \right.$$

One may compute directly how to go from one line to the next through writing the squares and working out the result. One may also proceed "syntaxically" using the following grammar:

σ-1) Write the initial natural transformation with a <u>left</u> arrow: .... $\underline{\mathbf{V}}\,\underline{\mathbf{T}} \longleftarrow \underline{\mathbf{U}}\,\underline{\mathbf{S}}$

σ-2) Consider this writing as a five-letter word, obtained as the concatenation of three words as follows: .... $\boxed{\underline{\mathbf{V}}}\,\boxed{\underline{\mathbf{T}} \longleftarrow \underline{\mathbf{U}}}\,\boxed{\underline{\mathbf{S}}}$

σ-3) Replace the two outer factors by their <u>left</u>-adjoints: .... $\boxed{\underline{\mathbf{V}}^{\sigma}}\,\boxed{\underline{\mathbf{T}} \longleftarrow \underline{\mathbf{U}}}\,\boxed{\underline{\mathbf{S}}^{\sigma}}$

σ-4) Reverse the middle "diagram": ..... $\boxed{\underline{\mathbf{V}}^{\sigma}}\,\boxed{\underline{\mathbf{U}} \longrightarrow \underline{\mathbf{T}}}\,\boxed{\underline{\mathbf{S}}^{\sigma}}$

σ-5) Forget the formal factorisation in three words, and read the result as a diagram: . . . . $\underline{\mathbf{V}}^{\sigma}\,\underline{\mathbf{U}} \longrightarrow \underline{\mathbf{T}}\,\underline{\mathbf{S}}^{\sigma}$

In fact, we may describe a single "line-wise" rule on our five-letter words:

$$\left[\begin{array}{l}\textit{Looking at an arrow as "oriented" (}\text{i.e. }\textit{going left or}\\ \textit{going right) , we reverse "the central diagram" and we}\\ \textit{mark the outer terms with an upper script to indicate}\\ \textit{the former orientation of the arrows: } \sigma \textit{ for "left", } \delta \textit{ for}\\ \textit{"right".}\end{array}\right] \qquad \text{(flip-flop)}$$

Since we suppose that, when adjoints exist, $\underline{\mathbf{U}}^{\delta\sigma} = \underline{\mathbf{U}} = \underline{\mathbf{U}}^{\sigma\delta}$ (*see* the beginning of the section), and since the δ-rule and the σ-rule are inverse of each-other, one immediately checks that the rewriting rules δ-(1-5) and σ-(1-5) are inverse of each other. The Δ-sequence and the Σ-sequence fit nicely together to yield one doubly infinite sequence of relations, that we refer to as *the* (Δ-Σ)-*sequence*; this is shown in Figure 3 where going upwards uses the σ-rule and downwards the δ-rule. In order to produce all of the Δ-Σ-sequence using (flip-flop) (*i.e.* just one rule), (flip-flop) must be applied to both writings of any mathematical arrow, that is to $\underline{\mathbf{U}}\,\underline{\mathbf{S}} \longrightarrow \underline{\mathbf{V}}\,\underline{\mathbf{T}}$ and to $\underline{\mathbf{V}}\,\underline{\mathbf{T}} \longleftarrow \underline{\mathbf{U}}\,\underline{\mathbf{S}}$.

To each square commutative up to a natural transformation, we attach its Δ-Σ-sequence; it is the maximal segment of the doubly infinite Δ-Σ-sequence of Figure 3; for each particular square, its length depends on the availability of adjoints.

As a first example, there are the squares $\begin{array}{ccc} \cdot & \xrightarrow{F} & \cdot \\ {\scriptstyle \mathbf{1}}\downarrow & =\!\nearrow & \downarrow{\scriptstyle \mathbf{1}} \\ \cdot & \xrightarrow[F]{} & \cdot \end{array}$ and $\begin{array}{ccc} \cdot & \xrightarrow{F} & \cdot \\ {\scriptstyle \mathbf{1}}\downarrow & =\!\swarrow & \downarrow{\scriptstyle \mathbf{1}} \\ \cdot & \xrightarrow[F]{} & \cdot \end{array}$ Their Δ-Σ-sequences are essentially the same, and are trivial since they are made of the identity transformations

$$\cdots \quad F^{\sigma} \longrightarrow F^{\sigma}, \quad F \longrightarrow F, \quad F^{\delta} \longrightarrow F^{\delta}, \quad \cdots$$

A richer example is given each time a pair of adjoint functors is given, say $F \uparrow\dashv\downarrow U$; if η is the unit and ϵ is the co-unit, we have $\begin{array}{ccc} \cdot & \xrightarrow{F} & \cdot \\ {\scriptstyle \mathbf{1}}\downarrow & \eta\nearrow & \downarrow{\scriptstyle U} \\ \cdot & \xrightarrow[\mathbf{1}]{} & \cdot \end{array}$ and $\begin{array}{ccc} \cdot & \xrightarrow{U} & \cdot \\ {\scriptstyle \mathbf{1}}\downarrow & \epsilon\swarrow & \downarrow{\scriptstyle F} \\ \cdot & \xrightarrow[\mathbf{1}]{} & \cdot \end{array}$ Their Δ-Σ-sequences are given in Figure 4 (part (a) and part (b) respectively). They are perfectly symetrical, and, given any one line in one of these two Δ-Σ-sequences, the rest of the sequence is totally determined (*i.e.* the "central lines" $1\,1 \longleftarrow F\,U$ and $1\,1 \rightarrow U\,F$ play no particular role).

$$
\begin{array}{c r r c l c}
 & \vdots & & \vdots & & \\
 & +5) & \underline{\mathbf{T}}^{\sigma\sigma}\,\underline{\mathbf{S}}^{\sigma\sigma\sigma} & \longleftarrow & \underline{\mathbf{V}}^{\sigma\sigma\sigma}\,\underline{\mathbf{U}}^{\sigma\sigma} & \\
 & +4) & \underline{\mathbf{U}}^{\sigma\sigma}\,\underline{\mathbf{S}}^{\sigma\sigma} & \longrightarrow & \underline{\mathbf{V}}^{\sigma\sigma}\,\underline{\mathbf{T}}^{\sigma\sigma} & \\
 & +3) & \underline{\mathbf{U}}^{\sigma}\,\underline{\mathbf{V}}^{\sigma\sigma} & \longleftarrow & \underline{\mathbf{S}}^{\sigma\sigma}\,\underline{\mathbf{T}}^{\sigma} & \\
 & +2) & \underline{\mathbf{T}}^{\sigma}\,\underline{\mathbf{V}}^{\sigma} & \longrightarrow & \underline{\mathbf{S}}^{\sigma}\,\underline{\mathbf{U}}^{\sigma} & \\
\uparrow & +1) & \underline{\mathbf{T}}\,\underline{\mathbf{S}}^{\sigma} & \longleftarrow & \underline{\mathbf{V}}^{\sigma}\,\underline{\mathbf{U}} & \downarrow \\
\sigma\text{-rule} & 0) & \underline{\mathbf{U}}\,\underline{\mathbf{S}} & \longrightarrow & \underline{\mathbf{V}}\,\underline{\mathbf{T}} & \delta\text{-rule} \\
 & -1) & \underline{\mathbf{U}}^{\delta}\,\underline{\mathbf{V}} & \longleftarrow & \underline{\mathbf{S}}\,\underline{\mathbf{T}}^{\delta} & \\
 & -2) & \underline{\mathbf{T}}^{\delta}\,\underline{\mathbf{V}}^{\delta} & \longrightarrow & \underline{\mathbf{S}}^{\delta}\,\underline{\mathbf{U}}^{\delta} & \\
 & -3) & \underline{\mathbf{T}}^{\delta\delta}\,\underline{\mathbf{S}}^{\delta} & \longleftarrow & \underline{\mathbf{U}}^{\delta\delta}\,\underline{\mathbf{V}}^{\delta} & \\
 & -4) & \underline{\mathbf{U}}^{\delta\delta}\,\underline{\mathbf{S}}^{\delta\delta} & \longrightarrow & \underline{\mathbf{V}}^{\delta\delta}\,\underline{\mathbf{T}}^{\delta\delta} & \\
 & -5) & \underline{\mathbf{U}}^{\delta\delta\delta}\,\underline{\mathbf{V}}^{\delta\delta} & \longleftarrow & \underline{\mathbf{S}}^{\delta\delta}\,\underline{\mathbf{T}}^{\delta\delta\delta} & \\
 & \vdots & & \vdots & &
\end{array}
$$

Figure 3: The $(\Delta\text{-}\Sigma)$-sequence of a square

1.4 Adjointness and ordered sets (AOS)

NT002e.txt This subsection mentions a few elementary facts, about functors between ordered sets, to be refered to in the course of this work.

[AOS-1]: Given $F \dashv U$, $(A,<) \underset{F}{\overset{U}{\rightleftarrows}} (X,<)$, we have $F\,U\,F = F$ and $U\,F\,U = U$. Given that $(A,<)$ and $(X,<)$ are Inf-complete, any given functor $U : (A,<) \to (X,<)$ has a left-adjoint if and only if U is inf-compatible, in which case the left-adjoint is given through $F(x) = \mathrm{Inf}\{a' : x < Ua'\}$. And given that $(A,<)$ and $(X,<)$ are sup-complete, any given functor $F : (X,<) \to (A,<)$ has a right-adjoint if and only if it is sup-compatible, in which case the right-adjoint is given through $Ua = \mathrm{Sup}\{x : Fx < a\}$.

3: Express the adjunction through FUF=F and UFU=U and ...

[AOS-2]: Given functors $(A,<) \underset{F}{\overset{U}{\rightleftarrows}} (X,<)$, with $F \dashv U$, $(A,<)$ is algebraic over $(X,<)$ if and only if $F\,U = \underline{\mathbf{Id}}_A$.

[AOS-3]: Given two functors $F_1, F_2 : \underline{\mathbf{Ens}} \to \underline{\mathbf{Ord}}$, such that for any set X, F_1X and F_2X are Inf-complete (in particular, for each X, F_2X has a supremum, which is the infimum on the empty family of elements of F_2X), and given a natural transformation $\lambda : F_1 \to F_2$ such that, for each set X, the λ_X component

$\vdots$		$\vdots$		$\vdots$		$\vdots$
$1^{\sigma\sigma}\, F^{\sigma\sigma}$	$\longrightarrow$	$1^{\sigma\sigma}\, F^{\sigma\sigma}$	identity	$F^{\sigma\sigma}\, 1^{\sigma\sigma\sigma}$	$\longleftarrow$	$F^{\sigma\sigma}\, 1^{\sigma\sigma}$
$1^{\sigma}\, 1^{\sigma\sigma}$	$\longleftarrow$	$F^{\sigma\sigma}\, F^{\sigma}$	counit - unit	$1^{\sigma\sigma}\, 1^{\sigma\sigma}$	$\longrightarrow$	$F^{\sigma}\, F^{\sigma\sigma}$
$F^{\sigma}\, 1^{\sigma}$	$\longrightarrow$	$F^{\sigma}\, 1^{\sigma\sigma}$	identity	$1^{\sigma}\, F^{\sigma}$	$\longleftarrow$	$1^{\sigma\sigma}\, F^{\sigma}$
$F\, F^{\sigma}$	$\longleftarrow$	$1^{\sigma}\, 1^{\sigma}$	unit - counit	$F^{\sigma}\, F$	$\longrightarrow$	$1^{\sigma}\, 1^{\sigma}$
$1^{\sigma}\, F$	$\longrightarrow$	$1\, F$	identity	$F\, 1^{\sigma}$	$\longleftarrow$	$F\, 1$
$1\, 1$	$\longleftarrow$	$F\, U$	counit - unit	$1\, 1$	$\longrightarrow$	$U\, F$
$U\, 1^{\delta}$	$\longrightarrow$	$U\, 1$	identity	$1^{\delta}\, U$	$\longleftarrow$	$1\, U$
$U^{\delta}\, U$	$\longleftarrow$	$1^{\delta}\, 1^{\delta}$	unit - counit	$U\, U^{\delta}$	$\longrightarrow$	$1^{\delta}\, 1^{\delta}$
$1^{\delta}\, U^{\delta}$	$\longrightarrow$	$1^{\delta\delta}\, U^{\delta}$	identity	$U^{\delta}\, 1^{\delta}$	$\longleftarrow$	$U^{\delta}\, 1^{\delta\delta}$
$1^{\delta\delta}\, 1^{\delta\delta}$	$\longleftarrow$	$U^{\delta}\, U^{\delta\delta}$	counit - unit	$1^{\delta\delta}\, 1^{\delta\delta}$	$\longrightarrow$	$U^{\delta\delta}\, U^{\delta}$
$U^{\delta\delta}\, 1^{\delta\delta\delta}$	$\longrightarrow$	$U^{\delta\delta}\, 1^{\delta\delta}$	identity	$1^{\delta\delta\delta}\, U^{\delta\delta}$	$\longleftarrow$	$1^{\delta\delta}\, U^{\delta\delta}$
$\vdots$		$\vdots$		$\vdots$		$\vdots$
	(a)				(b)	

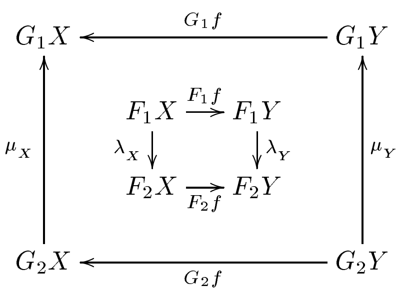
Figure 4: The $(\Delta$-$\Sigma)$-sequences for a pair of adjoint functors

is constant, one might be tempted to believe that each λ_X admits a left adjoint "because it is then Inf-compatible". However, the last statement is false in general because of the empty family. Indeed, let us suppose that, for all X, $\lambda_X a = b$. Then, for a family $\{a_i\}_{i\in I}$ of elements of X,

$$\lambda_X \operatorname{Inf}\{a_i\}_{i\in I} = b \quad \text{and} \quad \operatorname{Inf}\{\lambda_X a_i\}_{i\in I} = \begin{cases} \operatorname{Sup} F_2X & \text{if } I = \emptyset \\ b & \text{if } I \neq \emptyset \end{cases}$$

Therefore, *a constant componentwise natural transformation* $\lambda : F_1 \to F_2$ *satisfies "componentwise adjointness" if and only if, for any set* X *and all* $a \in X$, $\lambda_X a = \operatorname{Sup} F_2X$. In this case, the left-adjoint $\lambda^\sigma_X : F_2X \to F_1X$ is given, according to [AOS-1], by $\lambda^\sigma_X x = \operatorname{Inf} F_1X$. This applies in particular when F_1, F_2 are iterates of $\mathcal{P}$.

[AOS-4] "*The principle of adjoint naturality*" ("pan"): Let us be given: (1) two functors F_1, F_2 : **Ens** $\to$ **Ord**, with same variance, and two functors G_1, G_2 : **Ens** $\to$ **Ord**, with same object-value as F_1 and F_2 but of opposite variance, such that for each map f, $G_i f$ is left-adjoint (resp. right-adjoint) to $F_i f$ $(i = 1, 2)$; (2) a natural transformation $\lambda : F_1 \to F_2$ such that for each set X, λ_X has a left-adjoint (resp. right-adjoint) μ_X. Then $\mu = \{\mu_X\}_{X \text{ a set}}$ defines a natural transformation $G_2 \to G_1$.



(Diagram for the "pan")

Indeed, we have (for example, in the case of left-adjoints, F_1 covariant)

$$\left\{\begin{array}{lll} 1.1 & a < (\lambda_Y F_1 f)\,(G_1 f\,\mu_Y)\,a & \text{(The unity of } (G_1 f\,\mu_Y) \dashv (\lambda_Y F_1 f)) \\ \hline 1.2 & a < (F_2 f\,\lambda_X)\,(G_1 f\,\mu_Y)\,a & \text{(1.1 and the naturality of } \lambda) \\ \hline 1.3 & (\mu_X G_2 f)\,a < (G_1 f\,\mu_Y)\,a & \text{(1.2 and } (\mu_X G_2 f) \dashv (F_2 f\,\lambda_X)) \end{array}\right.$$

$$\left\{\begin{array}{lll} 2.1 & a < (F_2 f\,\lambda_X)\,(\mu_X G_2 f)\,a & \text{(The unity of } (\mu_X G_2 f) \dashv (F_2 f\,\lambda_X)) \\ \hline 2.2 & a < (\lambda_Y F_1 f)\,(\mu_X G_2 f)\,a & \text{(2.1 and the naturality of } \lambda) \\ \hline 2.3 & (G_1 f\,\mu_Y)\,a < (\mu_X G_2 f)\,a & \text{(2.2 and } (G_1 f\,\mu_Y) \dashv (\lambda_Y)\,F_1 f) \end{array}\right.$$

and the result follows immediately from 1.3 and 2.3. The other cases are treated similarly.

REMARK:: To say that $\lambda : F_1 \to F_2$ satisfies cw-adjointness means that for each set X, λ_X admits an adjoint (say a left-adjoint) μ_X, but this does not imply that $\mu = \{\mu_X\}_{X \text{ a set}}$ defines a natural transformation $F_2 \to F_1$, *i.e.* this transformation μ may fail to be natural. For example, for each set X, $\exists_{\supset} X \underset{\mathfrak{f}_X}{\overset{\mathfrak{e}_X}{\rightleftarrows}} \exists_{\supset} X$ with $\mathfrak{f}_X \dashv \mathfrak{e}_X$, where $\mathfrak{e}$ is the empty-set transformation ($\mathfrak{e}_X : A \mapsto \emptyset$), and $\mathfrak{f}$ is the full-set transformation ($\mathfrak{f}_X : A \mapsto X$). It is easily checked that $\mathfrak{e}$ is natural and $\mathfrak{f}$ is not natural from $\exists_{\supset}$ to itself. However, $\mathfrak{f}$ is natural $\underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\supset}$ (adjoint naturality).

The principle of adjoint naturality says that μ is natural from the left-adjoint of F_2 to the left-adjoint of F_1.

[AOS-5]: If $\begin{array}{ccc} \cdot & \xrightarrow{T} & \cdot \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle V} \\ \cdot & \xrightarrow[U]{} & \cdot \end{array}$ is a square of functors between ordered sets, with $U\,S < V\,T$, its $(\Delta\text{-}\Sigma)$-sequence (*see* Figure 3, page 14) is sequence (a) below, while if this square is commutative, we also have $V\,T < U\,S$, for which transformation the $(\Delta\text{-}\Sigma)$-sequence is sequence (b) below.

$$
\begin{array}{rcl}
 & \vdots & \\
U^{\sigma\sigma}\,S^{\sigma\sigma} & < & V^{\sigma\sigma}\,T^{\sigma\sigma} \\
U^{\sigma}\,V^{\sigma\sigma} & > & S^{\sigma\sigma}\,T^{\sigma} \\
T^{\sigma}\,V^{\sigma} & < & S^{\sigma}\,U^{\sigma} \\
T\,S^{\sigma} & > & V^{\sigma}\,U \\
U\,S & < & V\,T \\
U^{\delta}\,V & > & S\,T^{\delta} \\
T^{\delta}\,V^{\delta} & < & S^{\delta}\,U^{\delta} \\
T^{\delta\delta}\,S^{\delta} & > & V^{\delta}\,U^{\delta\delta} \\
U^{\delta\delta}\,S^{\delta\delta} & < & V^{\delta\delta}\,T^{\delta\delta} \\
 & \vdots & \\
 & (a) &
\end{array}
\qquad\qquad
\begin{array}{rcl}
 & \vdots & \\
V^{\sigma\sigma}\,T^{\sigma\sigma} & < & U^{\sigma\sigma}\,S^{\sigma\sigma} \\
V^{\sigma}\,U^{\sigma\sigma} & > & T^{\sigma\sigma}\,S^{\sigma} \\
S^{\sigma}\,U^{\sigma} & < & T^{\sigma}\,V^{\sigma} \\
S\,T^{\sigma} & > & U^{\sigma}\,V \\
V\,T & < & U\,S \\
V^{\delta}\,U & > & T\,S^{\delta} \\
S^{\delta}\,U^{\delta} & < & T^{\delta}\,V^{\delta} \\
S^{\delta\delta}\,T^{\delta} & > & U^{\delta}\,V^{\delta\delta} \\
V^{\delta\delta}\,T^{\delta\delta} & < & U^{\delta\delta}\,S^{\delta\delta} \\
 & \vdots & \\
 & (b) &
\end{array}
$$

These two sequences merge into the following formular for commutative squares of functors between ordered sets:

$$
\left.\begin{array}{c}
\vdots \\
V^{\sigma\sigma}\,T^{\sigma\sigma} = U^{\sigma\sigma}\,S^{\sigma\sigma} \\
U^{\sigma}\,V^{\sigma\sigma} > S^{\sigma\sigma}\,T^{\sigma} \text{ and } V^{\sigma}\,U^{\sigma\sigma} > T^{\sigma\sigma}\,S^{\sigma} \\
S^{\sigma}\,U^{\sigma} = T^{\sigma}\,V^{\sigma} \\
T\,S^{\sigma} > V^{\sigma}\,U \text{ and } S\,T^{\sigma} > U^{\sigma}\,V \\
V\,T = U\,S \\
U^{\delta}\,V > S\,T^{\delta} \text{ and } V^{\delta}\,U > T\,S^{\delta} \\
S^{\delta}\,U^{\delta} = T^{\delta}\,V^{\delta} \\
T^{\delta\delta}\,S^{\delta} > V^{\delta}\,U^{\delta\delta} \text{ and } S^{\delta\delta}\,T^{\delta} > U^{\delta}\,V^{\delta\delta} \\
V^{\delta\delta}\,T^{\delta\delta} = U^{\delta\delta}\,S^{\delta\delta} \\
\vdots
\end{array}\right] \qquad \text{(Delta-Sigma)}
$$

2 Traveling about the identity and $\mathcal{P}$

2.1 Between $\mathbf{Id}_{\mathbf{Ens}}$ and $\mathcal{P}$

NT003a.txt The lowest level for travelling between the $\mathcal{P}^i$'s is between $\mathcal{P}^0$ (the identity transformation) and $\mathcal{P}^1 = \mathcal{P}$.

4: Dans la version propre, pas oublier fichiers *.aux et *.toc pour inclure soulignements foncteurs (underline)

[A] There are no other endo-transformations of the identity endo-functor on **Ens** other than the identity endo-transformation, for if there were one such transformation λ, with one component λ_X not the identity, say $x_0 \neq \lambda_X x_0$, for some $x_0 \in X$, then $f : \{\mathbf{0}\} \to X$ being $\mathbf{0} \mapsto x_0$, the square $\begin{array}{ccc} \{\mathbf{0}\} & \xrightarrow{f} & X \\ {\scriptstyle\lambda_{\{0\}}}\Vert & & \downarrow{\scriptstyle\lambda_X} \\ \{\mathbf{0}\} & \xrightarrow[f]{} & X \end{array}$ would not be commutative.

[B] There are no transformations of the form $\mathcal{P} \to \underline{\mathbf{Id}_{\mathbf{Ens}}}$ because of the impossibility to have an $\emptyset$-component $\mathcal{P}\emptyset \to \emptyset$.

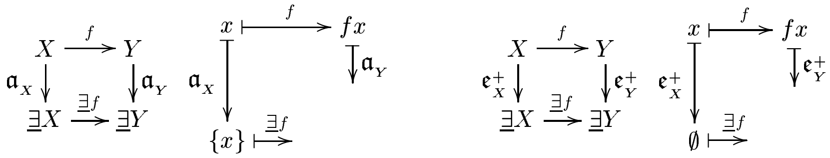
[C] Therefore, all that remains to consider is the case of transformations $\underline{\mathbf{Id}_{\mathbf{Ens}}} \to \underline{\exists}$. The case $\underline{\mathbf{Id}_{\mathbf{Ens}}} \to \underline{\forall}$ is obtained through conjugation by the natural equivalence $\nu : \underline{\exists} \to \underline{\forall}$. Of course, the $\emptyset$-component of such a transformation is the empty application $\emptyset : \emptyset \to \{\emptyset, \{\emptyset\}\}$.

There are two obvious natural transformations $\underline{\mathbf{Id}_{\mathbf{Ens}}} \to \underline{\exists}$, namely the *atom* or *minatom transformation*, written $\mathfrak{a}$, given by

$$\textit{for each non-empty set } X,\ \mathfrak{a}_X : X \to \mathcal{P}X : x \mapsto \{x\}, \qquad \text{(minatom)}$$

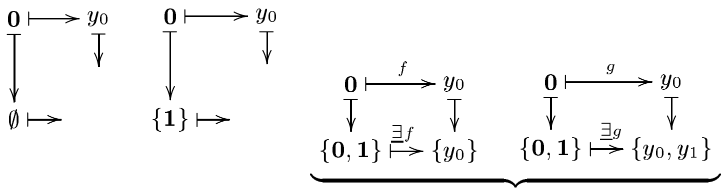
and the *empty-set transformation*, written $\mathfrak{e}^+$ (the + expresses that we go one level upward), given by

$$\textit{for each non-empty set } X,\ \mathfrak{e}^+_X : X \to \mathcal{P}X : x \mapsto \emptyset. \qquad \text{(empty)}$$



We show that there are no others. In what follows, λ shall mean a natural transformation other than $\mathfrak{e}$ and $\mathfrak{a}$; in particular, this means that there exists a non empty set Y with an element y such that $\lambda_Y y \neq \emptyset$. Unless otherwise stated, $X, Y, \ldots$ are non empty sets (because $\lambda_\emptyset$ is known).

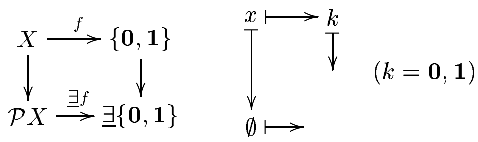
1) *The* $\{\mathbf{0},\mathbf{1}\}$*-component* $\lambda_{\{0,1\}}$ can only be the atom $\{\mathbf{0},\mathbf{1}\} \to \mathcal{P}\{\mathbf{0},\mathbf{1}\}$. First $\lambda_{\{0,1\}}\mathbf{0} = \{\mathbf{0}\}$, for the three other possibilities yield a contradiction:



(a) (b) (c)

(a) yields directly a contradiction through choosing a Y such that $\lambda_Y : Y \to \exists Y$ takes a value $y_O \neq \emptyset$; (b) yields a contradiction through choosing Y with at least two elements (for we may choose $y_1 = f\mathbf{1}$ different from $\lambda_Y f\mathbf{0}$; (c), if $f\mathbf{1} = f\mathbf{0} = g\mathbf{0} = y_0$, and $g\mathbf{1} = y_1 \neq y_0$, then $\lambda_Y y_0 = \{y_0, y_1\}$ and $\lambda_Y y_1 = \{\mathbf{0}\}$: contradiction. For similar reasons, we must have $\lambda_{\{0,1\}}\mathbf{1} = \{\mathbf{1}\}$

2) *The* X*-component for an arbitrary non-empty set* X: (2.a) For no set X is λ_X the constant map $X \to \mathcal{P}X$ with value the empty set:



(2.b) Let us now suppose that for a certain set X there exists an $x \in X$ such that $\lambda_X x = \{x_0, x_1, \cdots\}$. Let $f : X \to \mathbf{2}$ with $fx_0 = 0$, $fx_1 = \mathbf{1}$. Then the diagram below is not commutative :

$$\begin{array}{ccc} X & \xrightarrow{f} & \{\mathbf{0},\mathbf{1}\} \\ {\scriptstyle\lambda_X}\downarrow & & \downarrow{\scriptstyle\lambda_{\{\mathbf{0},\mathbf{1}\}}} \\ \underline{\exists}X & \xrightarrow{\underline{\exists}f} & \underline{\exists}\{\mathbf{0},\mathbf{1}\} \end{array} \qquad \begin{array}{ccc} x & \longmapsto & k \\ \downarrow & & \downarrow \\ \{x_0, x_1, \cdots\} & \longmapsto & \end{array} \quad (k = \mathbf{0}, \mathbf{1})$$

Therefore, it follows from (2.a) and (2.b) that $\lambda_X x$ is a one-point set for each $x \in X$. It remains to show that $\lambda_X x = \{x'\} \neq \{x\}$ is impossible. Indeed, if such a case would exist, through choosing f such that $fx = \mathbf{0}$, $fx' = \mathbf{1}$, we would contradict the naturality of λ:

$$\begin{array}{ccc} x & \longmapsto & \mathbf{0} \\ \downarrow & & \downarrow \\ \{x'\} & \longmapsto & \end{array}$$

We conclude from the above discussion that there are no natural transformations $\underline{\mathbf{Id}_{\mathbf{Ens}}} \to \underline{\exists}$, apart from $\mathfrak{e}^+$ and $\mathfrak{a}$.

Therefore, there are no other natural transformations $\underline{\mathbf{Id}_{\mathbf{Ens}}} \to \underline{\forall}$ apart from the *co-atom* or *maxatom transformation*, written $\mathfrak{z}$, given by

$$\textit{for each non-empty set } X,\ \mathfrak{z}_X : X \to \mathcal{P}X : x \mapsto X \backslash \{x\}, \qquad \text{(maxatom)}$$

and from the *full-set transformation*, written $\mathfrak{f}^+$, given by

$$\textit{for each non-empty set } X,\ \mathfrak{f}^+_X : X \to \mathcal{P}X : x \mapsto X. \qquad \text{(full)}$$

2.2 About $\mathcal{P}$

2.2.1 The covariant→covariant case

NT003b.txt Let us be given a natural transformation $\lambda : \underline{\exists} \to \underline{\exists}$, *i.e.* for each $f : X \to Y$,

$$\begin{array}{ccc} \underline{\exists}X & \xrightarrow{\underline{\exists}f} & \underline{\exists}Y \\ {\scriptstyle\lambda_X}\downarrow & & \downarrow{\scriptstyle\lambda_Y} \\ \underline{\exists}X & \xrightarrow[\underline{\exists}f]{} & \underline{\exists}Y \end{array} \qquad \lambda_Y\{fx : x \in A\} = \{fx : x \in \lambda_X A\} \qquad (1)$$

There are two "*extreme*" cases of natural transformations $\underline{\exists} \to \underline{\exists}$; there is on the one hand the identity transformation, and on the other hand the *empty-set transformation*, written $\mathfrak{e}$, the X-components of which are the $\overline{\emptyset}$, the constant maps with value $\emptyset \in \underline{\exists}X$. We show that these are the only two possibilities. The transformation $\mathfrak{e}^+ : \underline{\mathbf{Id}} \to \underline{\exists}$ (*see* the line labelled (empty) page 19) is recovered here as $\mathfrak{e}^+ = \mathfrak{e}\,\mathfrak{a}$.

The logical crux in describing all $\lambda : \underline{\exists} \to \underline{\exists}$ resides in the knowledge of $\lambda_{\mathbf{2}}$ (with $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$). If in (1) we set $Y = \mathbf{2}$ and $f = \chi_A$ (the characteristic function of some subset of A of some X), it reads

$$\lambda_{\mathbf{2}}\{\chi_A x : x \in A\} = \{\chi_A x : x \in \lambda_X A\} \qquad (2)$$

a relation that holds true for all X and all $A \subset X$. Setting $A = \emptyset$, (2) reads

$$\lambda_{\mathbf{2}}\emptyset = \begin{cases} \{\mathbf{0}\}, & \text{if } \lambda_X \emptyset \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

The left-hand side being constant, this means that $\lambda_X \emptyset$ (the value of λ_X at the *element* $\emptyset$ of $\underline{\exists}X$), for all sets X, is either always non empty or always empty. Now, $\lambda_\emptyset \emptyset = \emptyset$; therefore,$\lambda_X \emptyset = \emptyset$ for all X, and in particular, $\lambda_{\mathbf{2}}\emptyset = \emptyset$. If $N : \mathbf{2} \to \mathbf{2}$ is the negation, it follows from

$$\begin{array}{ccc} \begin{array}{ccc} \underline{\exists}\mathbf{2} & \xrightarrow{\underline{\exists}N} & \underline{\exists}\mathbf{2} \\ {\scriptstyle\lambda_{\mathbf{2}}}\downarrow & & \downarrow{\scriptstyle\lambda_{\mathbf{2}}} \\ \underline{\exists}\mathbf{2} & \xrightarrow[\underline{\exists}N]{} & \underline{\exists}\mathbf{2} \end{array} & \begin{array}{ccc} \{\mathbf{0},\mathbf{1}\} & \xmapsto{\underline{\exists}N} & \{\mathbf{0},\mathbf{1}\} \\ \downarrow & & \downarrow \\ \{\mathbf{0}\} & \xmapsto{\underline{\exists}N} & \end{array} & \begin{array}{ccc} \{\mathbf{0},\mathbf{1}\} & \xmapsto{\underline{\exists}N} & \{\mathbf{0},\mathbf{1}\} \\ \downarrow & & \downarrow \\ \{\mathbf{1}\} & \xmapsto{\underline{\exists}N} & \end{array} \\ \underline{\exists}N\lambda_{\mathbf{2}}A = \lambda_{\mathbf{2}}\underline{\exists}NA & & \\ (a) & (b) & (c) \end{array} \qquad (3)$$

that neither $\lambda_{\mathbf{2}}\{\mathbf{0},\mathbf{1}\} = \{\mathbf{0}\}$ nor $\lambda_{\mathbf{2}}\{\mathbf{0},\mathbf{1}\} = \{\mathbf{1}\}$. Thus

$$\textit{either } \lambda_{\mathbf{2}}\{\mathbf{0},\mathbf{1}\} = \emptyset, \qquad \textit{or } \lambda_{\mathbf{2}}\{\mathbf{0},\mathbf{1}\} = \{\mathbf{0},\mathbf{1}\}.$$

Let us now read (2) with $A = X = \{\emptyset\}$; this yields

$$\lambda_{\mathbf{2}}\{\mathbf{1}\} = \left\{\chi_{\{\emptyset\}} x : x \in \lambda_{\{\emptyset\}}\{\emptyset\}\right\} \qquad (4)$$

Since $\lambda_{\{\emptyset\}}\{\emptyset\}$ is either $\emptyset$ or $\{\emptyset\}$, we see (by $(3a)$) that

$$\begin{array}{ll} \textit{either } \lambda_{\mathbf{2}}\{\mathbf{1}\} = \emptyset, & \text{and then } \lambda_{\mathbf{2}}\{\mathbf{0}\} = \emptyset, \\ \textit{or } \lambda_{\mathbf{2}}\{\mathbf{1}\} = \{\mathbf{1}\}, & \text{and then } \lambda_{\mathbf{2}}\{\mathbf{0}\} = \{\mathbf{0}\}. \end{array}$$

Therefore, collecting this information shows that we are in one of the following situations:

$$(\alpha) : \begin{array}{ccc} \emptyset & \{\mathbf{0}\} & \{\mathbf{1}\} \\ \searrow & \downarrow & \swarrow \\ & \emptyset & \end{array} \qquad\qquad (\beta) : \begin{array}{ccc} \emptyset & \{\mathbf{0}\} & \{\mathbf{1}\} \\ \downarrow & \downarrow & \downarrow \\ \emptyset & \{\mathbf{0}\} & \{\mathbf{1}\} \end{array}$$

with $\lambda_{\mathbf{2}}\{\mathbf{0},\mathbf{1}\}$ either $\emptyset$ or $\{\mathbf{0},\mathbf{1}\}$.

If $A \neq \emptyset$, (2) yields

$$\lambda_{\mathbf{2}}\{\mathbf{1}\} = \big\{\chi_{_A} x : x \in \lambda_{_X} A\big\}. \tag{5}$$

If we now set $X = A = \{\mathbf{0},\mathbf{1}\}$, $\lambda_{\mathbf{2}}\{\mathbf{0},\mathbf{1}\} = \{\mathbf{0},\mathbf{1}\}$ implies $\lambda_{\mathbf{2}}\{\mathbf{1}\} = \{\mathbf{1}\}$; we are therefore in case (β), and thus $\lambda_{\mathbf{2}}$ is the identity on $\underline{\exists}\mathbf{2}$; on the other hand, $\lambda_{\mathbf{2}}\{\mathbf{0},\mathbf{1}\} = \emptyset$ implies $\lambda_{\mathbf{2}}\{\mathbf{1}\} = \emptyset$ which implies case (α), and thus $\lambda_{\mathbf{2}}$ is constant with value $\emptyset$. Thus, either $\lambda_{\mathbf{2}}$ is the identity on $\underline{\exists}\mathbf{2}$, or is $\ulcorner\emptyset\urcorner$.

Once $\lambda_{\mathbf{2}}$ is known, describing $\lambda_{_X} A$ for all X and all $A \subset X$ is straightforward. (1) *Let $\lambda_{\mathbf{2}}$ be the identity on* $\underline{\exists}\mathbf{2}$. If $A \neq \emptyset$, it follows from (5) that $\emptyset \neq \lambda_X A \subset A$. This in turn implies $\lambda_X A = A$ because, if $B = A \backslash \lambda_X A \neq \emptyset$, we would then have $\{\mathbf{0}\} = \underline{\exists}\chi_{_B}\lambda_X A = \lambda_{\mathbf{2}}\underline{\exists}\chi_{_B} A = \lambda_{\mathbf{2}}\{\mathbf{0},\mathbf{1}\} = \{\mathbf{0},\mathbf{1}\}$. Recall that it follows from (2) and $\lambda_{\mathbf{2}}\emptyset = \emptyset$ that $\lambda_{_X}\emptyset = \emptyset$. Therefore, for all X and all $A \subset X$, $\lambda_{_X} A = A$. Now (2) *let* $\lambda_{\mathbf{2}} = \ulcorner\emptyset\urcorner$ (the constant mapping with value $\emptyset$); then, it follows from (2) that $\lambda_{_X} A = \emptyset$ for all X and all $A \subset X$.

Since $\underline{\exists}$ and $\underline{\forall}$ are ν-conjugates, there is a one-to-one correspondence between the set of endo-transformations of $\underline{\exists}$ and the set of endo-transformations of $\underline{\forall}$, and through this, the *empty-set-transformation* $\mathfrak{e} : \underline{\exists} \to \underline{\exists}$ corresponds to the *full-set-transformation* $\mathfrak{f} : \underline{\forall} \to \underline{\forall}$, the X-components of which are the constant maps $\ulcorner X \urcorner$ with value X. The transfromation $\mathfrak{f}^+ : \underline{\mathbf{Id}} \to \underline{\forall}$ (*see* the line labelled (full) page 20) is recovered here as $\mathfrak{f}^+ = \mathfrak{f}\,\mathfrak{z}$.

In the same way, there are two natural transformations $\underline{\forall} \to \underline{\exists}$ and two natural transformations $\underline{\exists} \to \underline{\forall}$, namely the diagonals of the squares :

$$\begin{array}{ccc} \underline{\exists} & \overset{\nu}{\longleftrightarrow} & \underline{\forall} \\ {\scriptstyle \mathfrak{e}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}} \\ \underline{\exists} & \underset{\nu}{\longleftrightarrow} & \underline{\forall} \end{array} \qquad\qquad \begin{array}{ccc} \underline{\exists} & \overset{\nu}{\longleftrightarrow} & \underline{\forall} \\ {\scriptstyle Id}\big\downarrow & & \big\downarrow{\scriptstyle Id} \\ \underline{\exists} & \underset{\nu}{\longleftrightarrow} & \underline{\forall} \end{array}$$

All these transformations are given in Table 1.

Clearly, each of these natural transformations is componentwise functorial for some appropriate choice of spins (last column of Table 1-(a)). The spinned natural transformations λ satisfying componentwise adjointness are on the one hand (1), (3) (5) and (7) in Table 1-(a), since theses are either identities or identities followed by the componentwise auto-adjoint equivalence ν, and on the other hand (2), (4), (6) and (8) for exactly the spins shown in Table 1-(b). These transformations are exactly the componentwise constant transformations, with their "target spin" chosen so that the value is the supremum of the codomain (*see* [AOS-1]), and their left adjoints $\lambda^{\sigma}_{_X}$ obtained using [AOS-3]; depending on the spin of the domain functor, λ may be seen in two ways, each yielding

	X-Components	Values	cw-Functoriality
(1)	$(\underline{\mathbf{Id}}_{\exists})_X : \underline{\exists}X \to \underline{\exists}X$	$A \mapsto A$	For twice the same spin
(2)	$\mathfrak{e}_X : \underline{\exists}X \to \underline{\exists}X$	$A \mapsto \emptyset$	For all combinations of spins
(3)	$(\underline{\mathbf{Id}}_{\forall})_X : \underline{\forall}X \to \underline{\forall}X$	$A \mapsto A$	For twice the same spin
(4)	$\mathfrak{f}_X : \underline{\forall}X \to \underline{\forall}X$	$A \mapsto X$	For all combinations of spins
(5)	$\nu_X = (\nu\, \underline{\mathbf{Id}}_{\exists})_X : \underline{\exists}X \to \underline{\forall}X$	$A \mapsto X \backslash A$	For opposite spins
(6)	$\mathfrak{f}_X = (\nu\, \mathfrak{e})_X : \underline{\exists}X \to \underline{\forall}X$	$A \mapsto X$	For all combinations of spins
(7)	$\nu_X = (\nu\, \underline{\mathbf{Id}}_{\forall})_X : \underline{\forall}X \to \underline{\exists}X$	$A \mapsto X \backslash A$	For opposite spins
(8)	$\mathfrak{e}_X = (\nu\, \mathfrak{f})_X : \underline{\forall}X \to \underline{\exists}X$	$A \mapsto \emptyset$	For all combinations of spins

(a) Natural endo-transformations of covariant values of $\mathcal{P}$

	$\lambda : \diamond \to \circ$	X-components	The cw-left-adjoints.		Naturality?
(2)	$\mathfrak{e} : \underline{\exists}_{\subset} \to \underline{\exists}_{\supset}$ $\mathfrak{e} : \underline{\exists}_{\supset} \to \underline{\exists}_{\supset}$	$A \mapstochar\longrightarrow \emptyset$	$\mathfrak{e}_X : \underline{\exists}_{\subset}X \leftarrow \underline{\exists}_{\supset}X$ $\mathfrak{f}_X : \underline{\exists}_{\supset}X \leftarrow \underline{\exists}_{\supset}X$	$\emptyset \longleftarrow\mapstochar A$ $X \longleftarrow\mapstochar A$	yes no
(4)	$\mathfrak{f} : \underline{\forall}_{\subset} \to \underline{\forall}_{\subset}$ $\mathfrak{f} : \underline{\forall}_{\supset} \to \underline{\forall}_{\subset}$	$A \mapstochar\longrightarrow X$	$\mathfrak{e}_X : \underline{\forall}_{\subset}X \leftarrow \underline{\forall}_{\subset}X$ $\mathfrak{f}_X : \underline{\forall}_{\supset}X \leftarrow \underline{\forall}_{\subset}X$	$\emptyset \longleftarrow\mapstochar A$ $X \longleftarrow\mapstochar A$	no yes
(6)	$\mathfrak{f} : \underline{\exists}_{\subset} \to \underline{\forall}_{\subset}$ $\mathfrak{f} : \underline{\exists}_{\supset} \to \underline{\forall}_{\subset}$	$A \mapstochar\longrightarrow X$	$\mathfrak{e}_X : \underline{\exists}_{\subset}X \leftarrow \underline{\forall}_{\subset}X$ $\mathfrak{f}_X : \underline{\exists}_{\supset}X \leftarrow \underline{\forall}_{\subset}X$	$\emptyset \longleftarrow\mapstochar A$ $X \longleftarrow\mapstochar A$	yes no
(8)	$\mathfrak{e} : \underline{\forall}_{\subset} \to \underline{\exists}_{\supset}$ $\mathfrak{e} : \underline{\forall}_{\supset} \to \underline{\exists}_{\supset}$	$A \mapstochar\longrightarrow \emptyset$	$\mathfrak{e}_X : \underline{\forall}_{\supset}X \leftarrow \underline{\exists}_{\supset}X$ $\mathfrak{f}_X : \underline{\forall}_{\supset}X \leftarrow \underline{\exists}_{\supset}X$	$\emptyset \longleftarrow\mapstochar A$ $X \longleftarrow\mapstochar A$	no yes

(b) cw-adjointness for endo-transformations about covariant $\mathcal{P}$ that are not natural equivalences. None of them is cw-algebraic, and not all cw-left-adjoints are the components of natural transformation.

Table 1: Study of the natural endo-transformations of covariant values of $\mathcal{P}$

different componentwise left-adjoints; some of these componentwise adjoints define natural transformations, some do not. This is indicated at the right of the table. Clearly, none of (2), (4) (6) and (8) is cw-algebraic (*see* [AOS-2]). Thus, cw-algebraicity occurs only trivially about covariant $\mathcal{P}$, namely with the natural equivalences (1), (3), (5) and (7).

2.2.2 The contravariant→contravariant case

The four natural transformations $\lambda : \underline{\mathbf{C}} \to \underline{\mathbf{C}}$ of Proposition 1 (page 8) are easily described. For each $A \subset X$ we have the commutative diagram

$$\begin{array}{ccccc} X & & \underline{\mathbf{C}}X & \xrightarrow{\lambda_X} & \underline{\mathbf{C}}X \\ {\scriptstyle \chi_A}\downarrow & & {\scriptstyle \underline{\mathbf{C}}\chi_A}\uparrow & & \uparrow{\scriptstyle \underline{\mathbf{C}}\chi_A} \\ \mathbf{2} & & \underline{\mathbf{C}}\mathbf{2} & \xrightarrow[\lambda_{\mathbf{2}}]{} & \underline{\mathbf{C}}\mathbf{2} \end{array}$$

that is $\lambda_X A = \lambda_X \underline{\mathbf{C}}\chi_A\{\mathbf{1}\} = \underline{\mathbf{C}}\chi_A \lambda_{\mathbf{2}}\{\mathbf{1}\}$.

There are four possibilities, corresponding to the four possible values of $\lambda_{\mathbf{2}}\{\mathbf{1}\} \in \underline{\mathbf{C}}\mathbf{2} = \{\emptyset, \{\mathbf{1}\}, \{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}\}$. Table 2 (page 25) describes all these possibilities. In this table, $\mathfrak{e}$ is the *empty-set transformation* and $\mathfrak{f}$ the *full-set transformation* (Remark: by an abuse of notation, we use the same symbol "$\mathfrak{e}$" for $\mathfrak{e} : \underline{\exists} \to \underline{\exists}$, $\mathfrak{e} : \underline{\mathbf{C}} \to \underline{\mathbf{C}}$, $\mathfrak{e} : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\supset} \ldots$).

2.2.3 Linking results with the principle of adjoint naturality

The natural transformation $\underline{\mathbf{C}} \to \underline{\mathbf{C}}$ with cw-left-adjoints are given in Table 2 (page 25). Applying the "pan" (*see* [AOS-4], page 16) to each of these four cases yields :

$$\underbrace{\begin{array}{ccc} \underline{\mathbf{C}}_{\subset} & & \underline{\exists}_{\subset} \\ {\scriptstyle \mathfrak{e}}\downarrow & \overset{\text{pan}}{\longmapsto} & \uparrow{\scriptstyle \mathfrak{e}} \\ \underline{\mathbf{C}}_{\supset} & & \underline{\forall}_{\supset} \end{array}}_{(8)} \qquad \underbrace{\begin{array}{ccc} \underline{\mathbf{C}}_{\supset} & & \underline{\forall}_{\supset} \\ {\scriptstyle \mathfrak{e}}\downarrow & \overset{\text{pan}}{\longmapsto} & \uparrow{\scriptstyle \mathfrak{f}} \\ \underline{\mathbf{C}}_{\supset} & & \underline{\forall}_{\supset} \end{array}}_{(4)} \qquad \underbrace{\begin{array}{ccc} \underline{\mathbf{C}}_{\subset} & & \underline{\exists}_{\subset} \\ {\scriptstyle \mathfrak{f}}\downarrow & \overset{\text{pan}}{\longmapsto} & \uparrow{\scriptstyle \mathfrak{e}} \\ \underline{\mathbf{C}}_{\subset} & & \underline{\exists}_{\subset} \end{array}}_{(2)} \qquad \underbrace{\begin{array}{ccc} \underline{\mathbf{C}}_{\supset} & & \underline{\forall}_{\supset} \\ {\scriptstyle \mathfrak{f}}\downarrow & \overset{\text{pan}}{\longmapsto} & \uparrow{\scriptstyle \mathfrak{f}} \\ \underline{\mathbf{C}}_{\subset} & & \underline{\exists}_{\subset} \end{array}}_{(6)}$$

None of the resulting transformations is a left-adjoint, neither has any of them a left adjoint, as a quick look to Table 1 (page 23) shows. Numbers (8), (4), (2) and (6) above are refering to lines in Table 1-(a) where the resulting transformations are to be found.

$\lambda_{\mathbf{2}}\{\mathbf{1}\}$	X-COMPONENTS	VALUES	CW-FUNCTORIALITY
$\emptyset$	$\mathfrak{e}_X : \underline{\mathbf{C}}X \to \underline{\mathbf{C}}X$	$A \mapsto \emptyset$	for all combinations of spins
$\{\mathbf{1}\}$	$(\underline{\mathbf{Id}}_C)_X : \underline{\mathbf{C}}X \to \underline{\mathbf{C}}X$	$A \mapsto A$	for twice the same spin
$\{\mathbf{0}\}$	$\nu_X : \underline{\mathbf{C}}X \to \underline{\mathbf{C}}X$	$A \mapsto X \backslash A$	for opposite spins
$\{\mathbf{0}, \mathbf{1}\}$	$\mathfrak{f}_X : \underline{\mathbf{C}}X \to \underline{\mathbf{C}}X$	$A \mapsto X$	for all combinations of spins

(a) Natural endo-transformations of $\underline{\mathbf{C}}$

	$\lambda : \diamond \to \circ$	X-COMPONENTS	THE CW-LEFT-ADJOINTS.		NATURALITY?
$\emptyset$	$\mathfrak{e} : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\supset}$ $\mathfrak{e} : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\supset}$	$A \longmapsto \emptyset$	$\mathfrak{e}_X : \underline{\mathbf{C}}_{\subset}X \leftarrow \underline{\mathbf{C}}_{\supset}X$ $\mathfrak{f}_X : \underline{\mathbf{C}}_{\supset}X \leftarrow \underline{\mathbf{C}}_{\supset}X$	$\emptyset \longmapsfrom A$ $X \longmapsfrom A$	yes yes
$\{\mathbf{0}, \mathbf{1}\}$	$\mathfrak{f} : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\subset}$ $\mathfrak{f} : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\subset}$	$A \longmapsto X$	$\mathfrak{e}_X : \underline{\mathbf{C}}_{\subset}X \leftarrow \underline{\mathbf{C}}_{\subset}X$ $\mathfrak{f}_X : \underline{\mathbf{C}}_{\supset}X \leftarrow \underline{\mathbf{C}}_{\subset}X$	$\emptyset \longmapsfrom A$ $X \longmapsfrom A$	yes yes

(b) cw-adjointness for endo-transformations of $\underline{\mathbf{C}}$ that are not natural equivalences. In each case, the cw-adjoints are the components of a natural transformation.

Table 2: Study of the natural endo-transformations of $\underline{\mathbf{C}}$

Conversely, the natural transformations between two covariant forms of $\mathcal{P}X$ with cw-left-adjoints are given in Table 1. Applying the "pan" to the four cases where it is applicable yields:

$$\underbrace{\begin{array}{ccc} \underline{\exists}_{\supset} & & \underline{\mathbf{C}}_{\supset} \\ \mathfrak{e}\downarrow & \overset{\text{pan}}{\longmapsto} & \uparrow\mathfrak{f} \\ \underline{\exists}_{\supset} & & \underline{\mathbf{C}}_{\supset} \end{array}}_{\emptyset} \qquad \underbrace{\begin{array}{ccc} \underline{\forall}_{\subset} & & \underline{\mathbf{C}}_{\subset} \\ \mathfrak{f}\downarrow & \overset{\text{pan}}{\longmapsto} & \uparrow\mathfrak{e} \\ \underline{\forall}_{\subset} & & \underline{\mathbf{C}}_{\subset} \end{array}}_{\{\mathbf{0},\mathbf{1}\}} \qquad \underbrace{\begin{array}{ccc} \underline{\exists}_{\supset} & & \underline{\mathbf{C}}_{\supset} \\ \mathfrak{f}\downarrow & \overset{\text{pan}}{\longmapsto} & \uparrow\mathfrak{f} \\ \underline{\forall}_{\subset} & & \underline{\mathbf{C}}_{\subset} \end{array}}_{\{\mathbf{0},\mathbf{1}\}} \qquad \underbrace{\begin{array}{ccc} \underline{\forall}_{\subset} & & \underline{\mathbf{C}}_{\subset} \\ \mathfrak{e}\downarrow & \overset{\text{pan}}{\longmapsto} & \uparrow\mathfrak{e} \\ \underline{\exists}_{\supset} & & \underline{\mathbf{C}}_{\supset} \end{array}}_{\emptyset}$$

None of the resulting transformations has a left adjoint, but each of them is a left adjoint as Table 2 (page 25) shows. The labelling $\emptyset$, $\{\mathbf{0},\mathbf{1}\}$, ... *etc.* corresponds to lines in this table where the resulting transformations are to be found. One may observe that, in this way, the "pan" produces all natural adjoint transformations given in Table 2-(b).

3 Traveling from $\mathcal{P}$ to $\mathcal{P}^2$

3.1 The covariant→covariant case

3.1.1 Case $\exists \to \exists\,\exists$

CARACTERIZATION OF NATURAL TRANSFORMATIONS $\underline{\exists} \to \underline{\exists}\,\underline{\exists}$

NT004aa1.txt **A**– Let λ be a natural transformation $\underline{\exists} \to \underline{\exists}^2$; for each map $f : X \to Y$, we have the commutative square (a) below:

$$\begin{array}{ccc} \underline{\exists}X & \xrightarrow{\underline{\exists}f} & \underline{\exists}Y \\ \lambda_X\downarrow & & \downarrow\lambda_Y \\ \underline{\exists}^2X & \xrightarrow[\underline{\exists}^2 f]{} & \underline{\exists}^2Y \end{array} \qquad \begin{array}{ccc} \underline{\mathbf{B}}X & & \underline{\mathbf{B}}f(X) \\ \downarrow & & \downarrow \\ \underline{\exists}X & \xrightarrow{\underline{\exists}f} & \underline{\exists}f(X) \\ \lambda_X\downarrow & & \downarrow\lambda_{f(X)} \\ \underline{\exists}^2X & \xrightarrow[\underline{\exists}^2 f]{} & \underline{\exists}^2f(X) \end{array}$$

(a) (b)

If f is an *inclusion*, $\underline{\exists}f$, $\underline{\exists}^2 f$, ... are also clearly inclusions; consequently, λ_X is then the restriction of λ_Y to $\underline{\exists}X$ (and takes its values in $\underline{\exists}^2X \subset \underline{\exists}^2Y$), *i.e.* λ_X takes the same values at each point $A \in \underline{\exists}X \cap \underline{\exists}Y$ for all possible sets X, Y containing A as a subset, or said otherwise, $\lambda_X A$ depends only on A and not on the set X in which we consider A. Therefore, a natural transformation $\lambda : \underline{\exists} \to \underline{\exists}^2$ induces a rule associating with each subset A of any X containing

A a unique set $\lambda_X A$; so we have $\lambda_X A = \lambda_A A$. For each set X, $\lambda_X X$ is a subset of $\underline{\exists} X$ (*i.e.* an element of $\underline{\exists}^2 X$), which we write $\underline{\mathbf{B}}^{(\lambda)} X$. It is represented in Diagram (b) above, to which we now refer. Detailling, we have

$$\begin{array}{ccc} X & \xmapsto{\exists f} & f(X) \\ \downarrow & & \downarrow \\ \underline{\mathbf{B}}^{(\lambda)} X & \mapsto & \end{array}$$

i.e. $\underline{\mathbf{B}}^{(\lambda)} f(X) = \underline{\exists} f \, \underline{\mathbf{B}}^{(\lambda)} X$, and $\underline{\exists} f$ maps $A \in \underline{\mathbf{B}}^{(\lambda)} X$ to $f(A) \in \underline{\mathbf{B}}^{(\lambda)} f(X)$. Thus, for all sets X, the restriction of $\underline{\exists} f$ to $\underline{\mathbf{B}}^{(\lambda)} X$ is a surjective map

$$\underline{\mathbf{B}}^{(\lambda)} X \to \underline{\mathbf{B}}^{(\lambda)} f(X), \quad A \mapsto f(A).$$

We write $\underline{\mathbf{B}}^{(\lambda)} f$ the composite application $\underline{\mathbf{B}}^{(\lambda)} X \to \underline{\mathbf{B}}^{(\lambda)} f(X) \subset \underline{\mathbf{B}}^{(\lambda)} Y$. (Note: There is a possible ambiguity here; if it so happens that $A \subset X$ *and* $A \in X$, then fA has a meaning, and consequently, $\underline{\mathbf{B}}^{(\lambda)} fA$ may be read (adding braces) as $[\underline{\mathbf{B}}^{(\lambda)} f]A$ or as $\underline{\mathbf{B}}^{(\lambda)}[fA] \ldots$, which are generally *a priori* different.)

The construction

$$\begin{array}{ccc} X & & \underline{\mathbf{B}}^{(\lambda)} X \\ \downarrow{\scriptstyle f} & \longmapsto & \downarrow{\scriptstyle \underline{\mathbf{B}}^{(\lambda)} f} \\ Y & & \underline{\mathbf{B}}^{(\lambda)} Y \end{array}$$

is a *subfunctor* of $\underline{\exists}$ —a subfunctor $\underline{\mathbf{B}}$ of $\underline{\exists}$ is an endofunctor of $\underline{\mathbf{Ens}}$, associating with each set X a subset $\underline{\mathbf{B}}\, X$ of $\underline{\exists} X$ in such a way that the inclusions $\underline{\mathbf{B}}\, X \subset \underline{\exists} X$ are the components of a natural transformation:

$$X \xrightarrow{f} Y \qquad \longmapsto \qquad \begin{array}{ccc} \underline{\mathbf{B}}\, X & \xrightarrow{\underline{\mathbf{B}} f} & \underline{\mathbf{B}}\, Y \\ \downarrow & & \downarrow \\ \underline{\exists} X & \xrightarrow[\underline{\exists} f]{} & \underline{\exists} Y \end{array}$$

Conclusion: A natural transformation $\lambda : \underline{\exists} \to \underline{\exists}^2$ induces a subfunctor $\underline{\mathbf{B}}^{(\lambda)}$ of $\underline{\exists}$ which is given by $\underline{\mathbf{B}}^{(\lambda)} X = \lambda_X X$ and $\underline{\mathbf{B}}^{(\lambda)} f = \underline{\exists} f|_{\underline{\mathbf{B}}^{(\lambda)} X}$.

$$\lambda \longmapsto \underline{\mathbf{B}}^{(\lambda)} \qquad (\mathcal{R}1)$$

B– Conversely, let us be given a subfunctor $\underline{\mathbf{B}}$ of $\underline{\exists}$:

$$f : X \to Y \qquad \rightsquigarrow \qquad \begin{array}{ccc} \underline{\mathbf{B}} X & \xrightarrow[\underline{\mathbf{B}} f]{} & \underline{\mathbf{B}} Y \\ \downarrow & & \downarrow \\ \underline{\exists} X & \xrightarrow{\underline{\exists} f} & \underline{\exists} Y \end{array}$$

To each set X, $\underline{\mathbf{B}}$ associates a subset $\underline{\mathbf{B}}X$ of $\underline{\exists}X$, *i.e.* a point in $\underline{\exists}^2X$. Therefore, for each $A \subset X$, we have:

$$\begin{array}{ccc} A & \longmapsto & \underline{\mathbf{B}}A \\ {\scriptstyle \underline{\exists}X \ni} & & {\scriptstyle \in \underline{\exists}^2 A \subset \underline{\exists}^2 X} \end{array}$$

i.e. $\underline{\mathbf{B}}$ induces, for each set X, a mapping $\lambda_X^{(\underline{\mathbf{B}})} : \underline{\exists}X \to \underline{\exists}^2X$, namely the mapping given by $\lambda_X^{(\underline{\mathbf{B}})}A = \underline{\mathbf{B}}A$.

Requiring that $\lambda^{(\underline{\mathbf{B}})}$ be natural amounts to requiring that $\underline{\mathbf{B}}\,f(A) = \underline{\exists}f\,\underline{\mathbf{B}}A$:

$$f : X \to Y \quad \rightsquigarrow \quad \begin{array}{ccc} \underline{\exists}X & \xrightarrow{\underline{\exists}f} & \underline{\exists}Y \\ {\scriptstyle \lambda_X^{(\underline{\mathbf{B}})}}\downarrow & & \downarrow{\scriptstyle \lambda_Y^{(\underline{\mathbf{B}})}} \\ \underline{\exists}^2X & \xrightarrow{\underline{\exists}^2 f} & \underline{\exists}^2Y \end{array} \qquad \begin{array}{ccc} A & \longmapsto & f(A) \\ \downarrow & & \downarrow \\ \underline{\mathbf{B}}A & \longmapsto & \end{array}$$

And in fact we have $\underline{\mathbf{B}}\,f(A) = \underline{\exists}f\,\underline{\mathbf{B}}A$; indeed, since in $\underline{\mathbf{Ens}}$ surjections are exactly the mappings with a right inverse, all endofunctors in $\underline{\mathbf{Ens}}$ preserve surjections. Therefore, from a surjection $f : A \twoheadrightarrow f(A)$ we obtain the surjection $\underline{\mathbf{B}}f = \underline{\exists}f|_{\underline{\mathbf{B}}A} : \underline{\mathbf{B}}A \twoheadrightarrow \underline{\mathbf{B}}f(A)$. From the very definition of $\underline{\mathbf{B}}f$, it follows therefore that $\underline{\exists}f\,\underline{\mathbf{B}}\,A = \underline{\mathbf{B}}\,f(A)$. This proves the naturality of $\lambda^{(\underline{\mathbf{B}})}$.

5: Nous utilisons ici un caractère spécifiquement ensembliste. Quelle est l'extension possible aux Topos et aux univers algébriques ?

Conclusion: A subfunctor $\underline{\mathbf{B}}$ *of* $\underline{\exists}$ *induces a natural transformation* $\lambda^{(\underline{\mathbf{B}})} : \underline{\exists} \to \underline{\exists}^2$*, which is given by* $\lambda_X^{(\underline{\mathbf{B}})}A = \underline{\mathbf{B}}A$.

$$\underline{\mathbf{B}} \longmapsto \lambda^{(\underline{\mathbf{B}})} \qquad (\mathcal{R}2)$$

C– We have:

Proposition 2 *Rules* $\mathcal{R}1$ *and* $\mathcal{R}2$ *are inverse of each other:*

$$\mathrm{Nat}(\underline{\exists}, \underline{\exists}^2) \underset{\mathcal{R}_2}{\overset{\mathcal{R}_1}{\rightleftarrows}} \mathrm{Sub}\,\underline{\exists}$$

Indeed:

(1) $\lambda \xrightarrow{\mathcal{R}1} \underline{\mathbf{B}}^{(\lambda)} \xrightarrow{\mathcal{R}2} \lambda^{(\underline{\mathbf{B}}^{(\lambda)})} \ : \lambda_X^{(\underline{\mathbf{B}}^{(\lambda)})}A = \underline{\mathbf{B}}^{(\lambda)}A = \lambda_A\,A = \lambda_X\,A,$

i.e. $\lambda^{(\underline{\mathbf{B}}^{(\lambda)})} = \lambda$.

(2) $\underline{\mathbf{B}} \xrightarrow{\mathcal{R}2} \lambda^{(\underline{\mathbf{B}})} \xrightarrow{\mathcal{R}1} \underline{\mathbf{B}}^{(\lambda^{(\underline{\mathbf{B}})})} \ : \underline{\mathbf{B}}^{(\lambda^{(\underline{\mathbf{B}})})}\,X = \lambda_X^{(\underline{\mathbf{B}})}\,X = \underline{\mathbf{B}}\,X,$

i.e. $\underline{\mathbf{B}}^{(\lambda^{(\underline{\mathbf{B}})})} = \underline{\mathbf{B}}$.

Corollary *Subfunctors of* $\underline{\exists}$ *classify natural transformations* $\underline{\exists} \to \underline{\exists}^2$.

6: Donner une version interne de cette bijection dans le topos $\underline{\mathbf{Ens}}^{\underline{\mathbf{Ens}}}$ avec l'objet $\exists$

$$\frac{\underline{\exists} \longrightarrow \underline{\exists}^2}{\rightarrowtail \underline{\exists}} \; : \; \frac{\underline{\exists} \xrightarrow{\lambda} \underline{\exists}^2}{\underline{\mathbf{B}}^{(\lambda)} \rightarrowtail \underline{\exists}} \quad , \quad \frac{\underline{\exists} \xrightarrow{\lambda^{(\underline{\mathbf{B}})}} \underline{\exists}^2}{\underline{\mathbf{B}} \rightarrowtail \underline{\exists}}$$

Subfunctors of $\underline{\exists}$

NT004aa2.txt We shall need the following examples of subfunctors of $\underline{\exists}$.

(1) The functor transforming each diagram $X \xrightarrow{f} Y$ into $\emptyset \xrightarrow{\emptyset} \emptyset$; we write $\ulcorner\emptyset\urcorner$ for this functor. The functor transforming each diagram $X \xrightarrow{f} Y$ into $\{\emptyset\} \xrightarrow{\mathbf{Id}} \{\emptyset\}$; we write $\ulcorner\{\emptyset\}\urcorner$ for this functor.

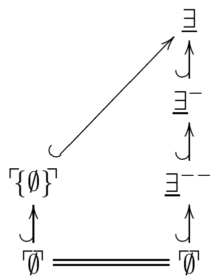
(2) The functors $\underline{\exists}^{--}$ and $\underline{\exists}^{-}$ given respectively by

$$(a)\ \underline{\exists}^{--}X = \{A \subset X; A \neq \emptyset\} \qquad (b)\ \underline{\exists}^{-}X = \begin{cases} \emptyset & \text{if } X = \emptyset \\ \underline{\exists}X & \text{if } X \neq \emptyset \end{cases}$$

Observe that, for each set X,

$$\begin{aligned} &\ulcorner\emptyset\urcorner X \subset \underline{\exists}^{--}X \subset \underline{\exists}^{-}X \subset \underline{\exists}X \\ &\ulcorner\emptyset\urcorner X \subset \ulcorner\{\emptyset\}\urcorner X \subset \underline{\exists}X \end{aligned}$$

whence we write the following diagram (that will be expanded into an infinite diagram presenting all subfunctors of $\underline{\exists}$ in Figure 5 (page 33)):



The description of all subfunctors of $\underline{\exists}$ rests on four elementary observations. Let $\underline{\mathbf{B}}$ be a subfunctor of $\underline{\exists}$.

Observation 1 *If* $\underline{\mathbf{B}} \neq \ulcorner\emptyset\urcorner$, *then for each set* $X \neq \emptyset$, $\underline{\mathbf{B}}\,X \neq \emptyset$.

There exists a $Y \neq \emptyset$ such that $\underline{\mathbf{B}}Y \neq \emptyset$, for if it were not the case, we would have $\underline{\mathbf{B}}\emptyset = \{\emptyset\}$, and the image by $\underline{\mathbf{B}}$ of the diagram $\emptyset \xrightarrow{\emptyset} Y$ would be a mapping $\{\emptyset\} \to \emptyset$, which is a contradiction.

Let $X \neq \emptyset$. There exists a mapping $Y \to X$, and its image is a mapping $\emptyset \neq \underline{\mathbf{B}}\,Y \to \underline{\mathbf{B}}\,X$. This implies $\underline{\mathbf{B}}\,X \neq \emptyset$.

Observation 2 *If* $\underline{\mathbf{B}} \neq \overline{\emptyset}$ *and if* $\emptyset \neq A \in \underline{\mathbf{B}}X$, *then for all* $X' \in \underline{\mathbf{B}}X$ *with* $0 < |A'| \leq |A|$, $A' \in \underline{\mathbf{B}}X$.

Indeed, $0 < |A'| \leq |A|$ implies that there exists a surjection $A \twoheadrightarrow A'$. We extend this surjection to a mapping $f : X \to X$; clearly, $f(A) = A'$. The result follows from the calculation of $\underline{\mathbf{B}}\,f$ and from the naturality of the inclusions ($A' = \underline{\mathbf{B}}\,fA \in \underline{\mathbf{B}}X$):

$$X \xrightarrow{\ f\ } X \qquad \longmapsto \qquad \begin{array}{ccc} \underline{\mathbf{B}}\,X & \xrightarrow{\underline{\mathbf{B}}\,f} & \underline{\mathbf{B}}\,X \\ \downarrow & & \downarrow \\ \exists\,X & \xrightarrow{\exists\,f} & \exists\,X \end{array}$$

Observation 3 *For each* $\underline{\mathbf{B}} \neq \overline{\emptyset}$ *we can canonically associate to each* $X \neq \emptyset$ *a cardinal* $\mathfrak{b}_X$, *namely* $\mathfrak{b}_X = \sup\{|A|; A \in \underline{\mathbf{B}}X\}$. *We call* $\mathfrak{b}_X$ the bound of $\underline{\mathbf{B}}$ at X.

This results from observation 1.

It results from observation 2 that for a certain X, if $\mathfrak{b}_X = 0$, then $\underline{\mathbf{B}}X = \{\emptyset\}$; if $\mathfrak{b}_X \neq 0$, then $\underline{\mathbf{B}}X$ is one of the followings:

(a) $[0, \mathfrak{b}_X) = \{A \in \mathcal{P}X; |A| < \mathfrak{b}_X\}$;

(b) $(0, \mathfrak{b}_X) = \{A \in \mathcal{P}X; \emptyset \neq A \text{ and } |A| < \mathfrak{b}_X\}$;

(c) $[0, \mathfrak{b}_X] = \{A \in \mathcal{P}X; |A| \leq \mathfrak{b}_X\}$;

(d) $(0, \mathfrak{b}_X] = \{A \in \mathcal{P}X; \emptyset \neq A \text{ et } |A| \leq \mathfrak{b}_X\}$.

Of course, when X runs through all non-empty sets, $\mathfrak{b}_X$ may *a priori* not always correspond to the same of the four definitions (a)-(d).

Observation 4 *If* $\mathfrak{b}_Y \leq |X|$, *then* $\mathfrak{b}_Y \leq \mathfrak{b}_X$; *if* $|Y| < \mathfrak{b}_X$, *then* $\mathfrak{b}_Y = |Y|$.

(a) Let $\mathfrak{b}_Y \leq |X|$. Let $A \in \underline{\mathbf{B}}Y$. Then, there exists an injection $A \rightarrowtail X$, and we extend it into a mapping $Y \to X$. Clearly, $|A| = |f(A)|$. The naturality of the inclusions implies then that $f(A) \in \underline{\mathbf{B}}X$, and therefore that $\mathfrak{b}_Y \leq \mathfrak{b}_X$.

$$\begin{array}{ccc} \underline{\mathbf{B}}\,Y & \xrightarrow{\underline{\mathbf{B}}\,f} & \underline{\mathbf{B}}\,X \\ \downarrow & & \downarrow \\ \exists\,Y & \xrightarrow{\exists\,f} & \exists\,X \end{array}$$

(b) Let $|Y| \leq \mathfrak{b}_X$. There exists then an $A \in \underline{\mathbf{B}}X$ and a surjection $A \twoheadrightarrow Y$ that we extend to a mapping $f : X \to Y$. Since $f(A) = Y$, the naturality of the inclusions implies $Y \in \underline{\mathbf{B}}Y$, and hence that $\mathfrak{b}_Y = |Y|$.

We say that a subfunctor $\underline{\mathbf{B}}$ of $\exists$ is *bounded* when there exists a cardinal $\mathfrak{b}$ such that for all sets X and for all $A \in \underline{\mathbf{B}}X$, $|A| \leq \mathfrak{b}$. If $\underline{\mathbf{B}}$ is bounded, the bounds $\mathfrak{b}_X$ are bounded, and conversely, if the bounds $\mathfrak{b}_X$ ($X \neq \emptyset$) form a bounded set of cardinals, $\underline{\mathbf{B}}$ is bounded. We call $\inf\{\mathfrak{b}; \forall X, \forall A \in \underline{\mathbf{B}}X, |A| \leq \mathfrak{b}\}$ *the bound of* $\underline{\mathbf{B}}$.

Proposition A *The unbounded subfunctors of* $\underline{\exists}$ *are* $\underline{\exists}^{--}$, $\underline{\exists}^{-}$ *and* $\underline{\exists}$.

DEMONSTRATION :

Let $\underline{\mathbf{B}}$ be unbounded. It results from Observation (4b) that $|Y| = \mathfrak{b}_Y$ for all non empty $Y \neq \emptyset$. Therefore, $\underline{\mathbf{B}}Y$ contains all non empty subsets of Y with cardinal $< |Y|$ (*see* observation 2).

Let Z be such that $|Z| > |Y|$. There exists then an $A \in \underline{\mathbf{B}}Z$ with $|Y| \leq |A| < |Z|$, which implies that there exists a surjection $Z \twoheadrightarrow Y$ such that $f(A) = Y$. It follows then from the naturality of the inclusions that $Y \in \underline{\mathbf{B}}Y$.

Thus, for each $X \neq \emptyset$, $\underline{\mathbf{B}}X$ contains $\{A \in \underline{\exists}X \ ; \ A \neq \emptyset\}$. Therfore, we have:

- If $X \neq \emptyset$, (a) $\underline{\mathbf{B}}X = \{A \in \underline{\exists}X; A \neq \emptyset\}$ or (b) $\underline{\mathbf{B}}X = \underline{\exists}X$;
- If $X = \emptyset$, (c) $\underline{\mathbf{B}}\emptyset = \emptyset$ or (d) $\underline{\mathbf{B}}\emptyset = \{\emptyset\}$.

However, (a) and (d) are simultaneously impossible because of the naturality of the inclusions,

$$\emptyset \to X \qquad \rightsquigarrow \qquad \begin{array}{ccc} \{\emptyset\} & \overset{?}{\dashrightarrow} & \{A \subset X; A \neq \emptyset\} \\ \downarrow & & \downarrow \\ \{\emptyset\} & \xrightarrow{\underline{\exists}\emptyset} & \underline{\exists}X \end{array} \qquad (\emptyset \mapsto \emptyset)$$

while (a)-(c) yields $\underline{\exists}^{--}$, (b)-(c) yields $\underline{\exists}^{-}$ and (b)-(d) yields $\underline{\exists}$.

—— QED (Proposition A)

Notation: For each cardinal α, and for each set X, we write

$$\underline{\exists}^{--}_{<\alpha}X = \{A \in \underline{\exists}X \ ; \ A \neq \emptyset \text{ and } |A| < \alpha\}$$
$$\underline{\exists}^{-}_{<\alpha}X = \begin{cases} \emptyset & \text{if } X = \emptyset \\ \underline{\exists}_{<\alpha}X & \text{if } X \neq \emptyset \end{cases}$$
$$\underline{\exists}_{<\alpha}X = \{A \in \underline{\exists}X \ ; \ |A| < \alpha\}$$

We observe that $\underline{\exists}^{--}_{<\alpha} \subset \underline{\exists}^{-}_{<\alpha} \subset \underline{\exists}_{<\alpha}$. In a similar way, we define $\underline{\exists}^{--}_{\leq\alpha}$, $\underline{\exists}^{-}_{\leq\alpha}$, $\underline{\exists}_{\leq\alpha}$, replacing "$<$" by "$\leq$", and likely, we have $\underline{\exists}^{--}_{\leq\alpha} \subset \underline{\exists}^{-}_{\leq\alpha} \subset \underline{\exists}_{\leq\alpha}$.

Proposition B *(a) The bounded-by-0 subfunctors of* $\underline{\exists}$ *are* $\underline{\exists}^{--}_{\leq 0}$, $\underline{\exists}^{-}_{\leq 0}$ *and* $\underline{\exists}_{\leq 0}$.

$$\text{Remark:} \quad \begin{cases} \underline{\exists}^{--}_{\leq 0} = \ulcorner\emptyset\urcorner \\ \underline{\exists}^{-}_{\leq 0}X = \begin{cases} \{\emptyset\} & \text{for all } X \neq \emptyset \\ \emptyset & \text{if } X = \emptyset \end{cases} \\ \underline{\exists}_{\leq 0} = \underline{\exists}^{--}_{<0} = \underline{\exists}^{-}_{<0} = \underline{\exists}_{<0} = \ulcorner\{\emptyset\}\urcorner \end{cases}$$

(b) If α *is an infinite successor cardinal, there are three bounded-by-*α *subfunctors of* $\underline{\exists}$*, namely* $\underline{\exists}^{--}_{\leq\alpha}$, $\underline{\exists}^{-}_{\leq\alpha}$ *and* $\underline{\exists}_{\leq\alpha}$.

(c) If α is a limit cardinal, there are six bounded-by-α subfunctors of $\underline{\exists}$, namely $\underline{\exists}^{--}_{<\alpha}$, $\underline{\exists}^{-}_{<\alpha}$, $\underline{\exists}_{<\alpha}$, and $\underline{\exists}^{--}_{\leq\alpha}$, $\underline{\exists}^{-}_{\leq\alpha}$, $\underline{\exists}_{\leq\alpha}$.

DEMONSTRATION :

(a) It is clear that $\ulcorner\emptyset\urcorner$ is bounded by 0. Therfore, we suppose that $\underline{\mathbf{B}} \neq \emptyset$. It results from Observation 1 that $\underline{\mathbf{B}}X \neq \{\emptyset\}$ for all $X \neq \emptyset$; therefore, the value of $\underline{\mathbf{B}}$ at $\emptyset$ is either $\emptyset$ or $\{\emptyset\}$, and both values define a subfunctor of $\underline{\exists}$ bounded by 0. There are therefore three subfunctors of $\underline{\exists}$ bounded by 0, namely:

$$\begin{array}{ll} \ulcorner\emptyset\urcorner & \text{this is } \underline{\exists}^{--}_{\leq 0} \\ X \mapsto \begin{cases} \emptyset & \text{if } X = \emptyset \\ \{\emptyset\} & \text{if } X \neq \emptyset \end{cases} & \text{this is } \underline{\exists}^{-}_{\leq 0} \\ X \mapsto \{A \subset X : |A| \leq 0\} & \text{this is } \underline{\exists}_{\leq 0} \end{array}$$

(b) and (c) are proved as in Proposition 1 and essentially result from Observation 3. —— QED (Proposition B)

We call a *borne* a bounded subfunctor of $\underline{\exists}$. Figure 5 presents all subfunctors of $\underline{\exists}$ with the inclusions linking them.

TABLE OF THE NATURAL TRANSFORMATIONS $\underline{\exists} \to \underline{\exists}^2$

For each $\underline{\mathbf{B}} \subset \underline{\exists}$, there is a natural transformation $\lambda^{(\underline{\mathbf{B}})} : \underline{\exists} \to \underline{\exists}^2$ as given by rule $\mathcal{R}2$ (28), namely

$$\lambda^{(\underline{\mathbf{B}})}_X A = \underline{\mathbf{B}}A$$

. This, together with the full description of all subfunctors of $\underline{\exists}$ given in Figure 5 yields Table 3 (page 34).

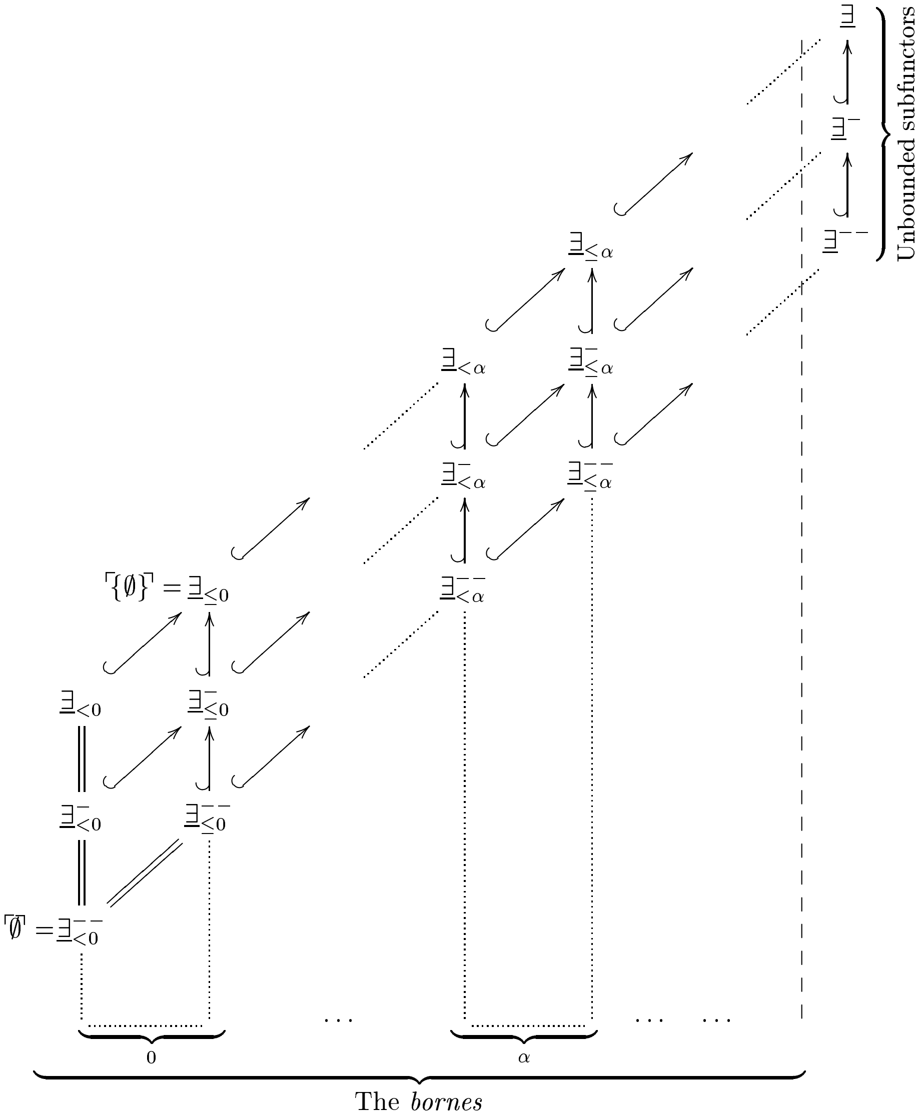


Figure 5: The subfunctors of $\underline{\exists}$. For each successor cardinal α and for $\alpha = 0$, there are three bounded-by-α subfunctors of $\underline{\exists}$, and for each limit cardinal, there are six of them. A bounded subfunctor of $\underline{\exists}$ is called a *borne*. There are three unbounded subfunctors of $\underline{\exists}$.

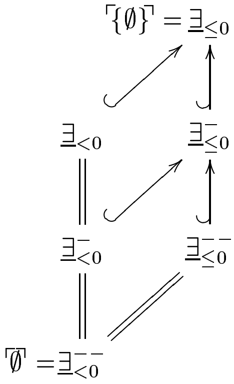
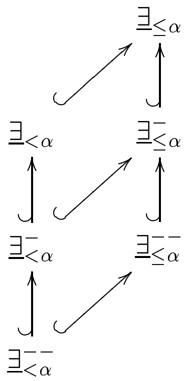
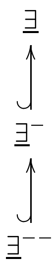
The subfunctors $\underline{\mathbf{B}}$	The natural transformations $\lambda^{(\underline{\mathbf{B}})}$
	$\lambda_X^{(\ulcorner\{\emptyset\}\urcorner)} A = \ulcorner\{\emptyset\}\urcorner$ $\lambda_X^{(\underline{\exists}^{--}_{\leq 0})} A = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{\emptyset\} & \text{if } A \neq \emptyset \end{cases}$ $\lambda_X^{(\ulcorner\emptyset\urcorner)} A = \emptyset$
	$\lambda_X^{(\underline{\exists}_{\leq \alpha})} A = \{U \in \underline{\exists} A \ ; \ \lvert U\rvert \leq \alpha\}$ $\lambda_X^{(\underline{\exists}_{< \alpha})} A = \{U \in \underline{\exists} A \ ; \ \lvert U\rvert < \alpha\}$ $\lambda_X^{(\underline{\exists}^{-}_{\leq \alpha})} A = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{U \in \underline{\exists} A \ ; \ \lvert U\rvert \leq \alpha\} & \text{if } A \neq \emptyset \end{cases}$ $\lambda_X^{(\underline{\exists}^{-}_{< \alpha})} A = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{U \in \underline{\exists} A \ ; \ \lvert U\rvert < \alpha\} & \text{if } A \neq \emptyset \end{cases}$ $\lambda_X^{(\underline{\exists}^{--}_{\leq \alpha})} A = \{U \in \underline{\exists} A \ ; \ U \neq \emptyset \text{ and } \lvert U\rvert \leq \alpha\}$ $\lambda_X^{(\underline{\exists}^{--}_{< \alpha})} A = \{U \in \underline{\exists} A \ ; \ U \neq \emptyset \text{ and } \lvert U\rvert < \alpha\}$
	$\lambda_X^{(\underline{\exists})} A = \{U \subset A\}$ $\lambda_X^{(\underline{\exists}^{-})} A = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \underline{\exists} A & \text{if } A \neq \emptyset \end{cases}$ $\lambda_X^{(\underline{\exists}^{--})} A = \{U \subset A ; U \neq \emptyset\}$

Table 3: Les transformations naturelles $\underline{\exists} \to \underline{\exists}^2$

3.1.2 Case $\exists \to \exists\,\forall$

3.1.3 Case $\exists \to \forall\,\exists$

3.1.4 Case $\exists \to \forall\,\forall$

3.1.5 Case $\exists \to C\,C$

3.1.6 Case $\forall \to \exists\,\exists$

3.1.7 Case $\forall \to \exists\,\forall$

3.1.8 Case $\forall \to \forall\,\exists$

3.1.9 Case $\forall \to \forall\,\forall$

3.1.10 Case $\forall \to C\,C$

3.2 The contravariant→contravariant case

3.2.1 The π-ψ-cube

NT004ba.txt

Among the $4\times 16 = 64$ natural transformations $\underline{\mathbf{C}} \to \underline{\mathbf{F}}$ of Proposition 1 (page 8), some have components which are functors (between ordered sets) for some choice of spin. Moreover, among these ones, some have an adjoint (left or right depending on the choice of the spin) and among these last ones, there is exactly one, up to within conjugation by ν, which, together with its adjoint, determines $\mathcal{P}X$ as algebraic over $\mathcal{P}^2X$. These facts will be proved in the course of section 3.2. We introduce this unique natural transformation, which we write π, its ν-conjugate, which we write ψ, in the line (pi-psi) below; π, ψ and their respective adjoints υ and δ respectively are presented in part 1 of Figure 6; the ν-conjugacy and adjointness properties are described in parts 2 and 3 of the figure, while part 4 tells that π and ψ are natural transformations from $\mathcal{P}$ to $\mathcal{P}^2$. The cube in part 4 of Figure 6 is called the "π-ψ-cube"; it summarizes naturalities and conjugations linking π and ψ.

$$\left.\begin{array}{l} \pi_X : \mathcal{P}X \to \mathcal{P}^2X : A \mapsto \big\{B \in \mathcal{P}X \ ; \ \forall x\big(x \in B \Rightarrow x \in A\big)\big\} = \pi_X A \\ \psi_X : \mathcal{P}X \to \mathcal{P}^2X : A \mapsto \big\{B \in \mathcal{P}X \ ; \ \exists x\big(x \in B \wedge x \in A\big)\big\} = \psi_X A \end{array}\right] \quad \text{(pi-psi)}$$

Let us point out a similarity of form between the formulæ expressing υ_X, ψ_X and $\underline{\exists} f$. In the case of ψ_X and $\underline{\exists} f$, starting from the formula for $\underline{\exists} fA$, we obtain

$$\underset{C}{\exists} fA = \mathfrak{a}_Y{}^* f^{**} \psi_X A$$

1)

$\mathcal{P} \downarrow \mathcal{P}^2$

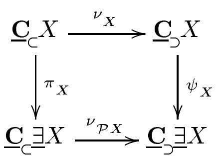
π_X	:	$\mathcal{P}X$	$\longrightarrow$	$\mathcal{P}^2X$
		A	$\longmapsto$	$\{B \in \mathcal{P}X;\ \forall x(x \in B \Rightarrow x \in A)\}$
ψ_X	:	$\mathcal{P}X$	$\longrightarrow$	$\mathcal{P}^2X$
		A	$\longmapsto$	$\{B \in \mathcal{P}X;\ \exists x(x \in B\ \wedge\ x \in A)\}$

$\mathcal{P} \uparrow \mathcal{P}^2$

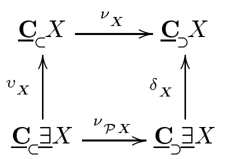
υ_X	:	$\mathcal{P}^2X$	$\longrightarrow$	$\mathcal{P}X$
		$\mathcal{A}$	$\longmapsto$	$\{x \in X\ ;\ \exists A(x \in A\ \wedge\ A \in \mathcal{A})\}$
δ_X	:	$\mathcal{P}^2X$	$\longrightarrow$	$\mathcal{P}X$
		$\mathcal{A}$	$\longmapsto$	$\{x \in X\ ;\ \forall A(x \in A \Rightarrow A \in \mathcal{A})\}$

$\pi_X A$ is the set of all subsets of X contained in A.
$\psi_X A$ is the set of all subsets of X meeting A.
$\upsilon_X \mathcal{A}$ is made of all x's belonging to at least one part belonging to $\mathcal{A}$.
$\delta_X \mathcal{A}$ is made of all x's not belonging to any part not belonging to $\mathcal{A}$.

2)



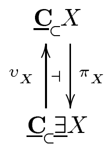
π and ψ are ν-conjugate.



υ and δ are ν-conjugate.

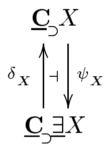
3)

$\dfrac{\upsilon_X \mathcal{A} \subset A}{\mathcal{A} \subset \pi_X A}$:



υ_X is left-adjoint to π_X.

$\dfrac{\delta_X \mathcal{A} \supset A}{\mathcal{A} \supset \psi_X A}$:



δ_X is left-adjoint to ψ_X.

4)

The π-ψ-cube

It is commutative.

π is a natural transformation $\underline{\mathbf{C}}_\subset \to \underline{\mathbf{C}}_\subset\exists$.
ψ is a natural transformation $\underline{\mathbf{C}}_\supset \to \underline{\mathbf{C}}_\supset\exists$.
ν is a natural transformation $\underline{\mathbf{C}} \to \underline{\mathbf{C}}$ as given. ν-components form pairs of adjoints.

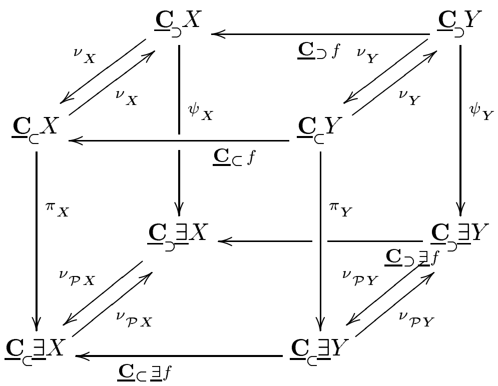


Figure 6: Some basic natural transformations between $\mathcal{P}$ and $\mathcal{PP}$

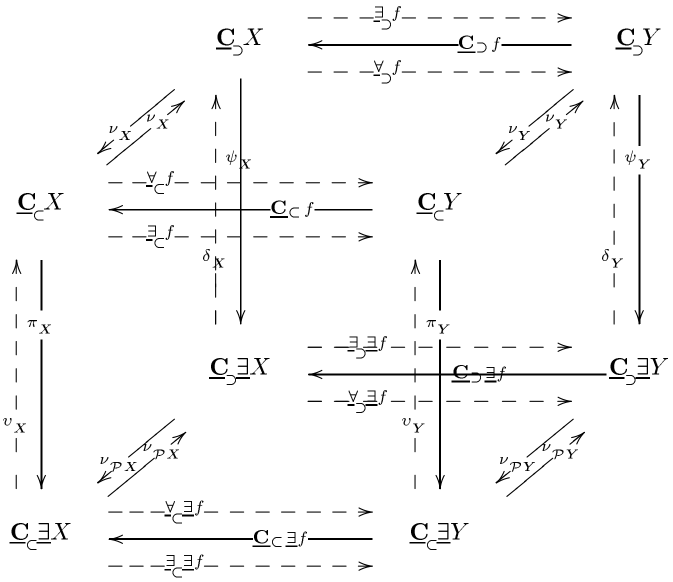


Figure 7: "In the neighbourhood" of the π-ψ-cube

where $\mathfrak{a}_Y$ is the minatom transformation (*see* the line labelled (minatom) on page 18).

In a similar way, we obtain

$$\underline{\forall}_{\subset} f A = \mathfrak{a}_Y{}^{*} f^{**} \pi_X A$$

3.2.2 The logical neighbourhood of the π-ψ-cube

NT004bb.txt The top-, bottom-, front-, rear-, and left-faces of the "π-ψ-cube" give us altogether essentially eight commutative squares of functors between ordered sets. Most functors in these squares have adjoints as summarized in the "cube" of Figure 7.

Applying [AOS-5] to each of these squares yields "*the logical neighbourhood*" of the π-ψ-cube as the concrete values taken by the formular for commutative squares of functors between ordered sets (*see* page 18) for these eight commutative squares. These formulars and the corresponding squares (faces) are given by the two-page figure on page 39. These formulæ are related to each other; for example, the left-hand side of line $B1$ is contained in line $b2$. When omitting redundant informations and connecting related ones, we obtain the following single formular, expressing the logical neighbouhood of the π-ψ-cube.

Proposition 3

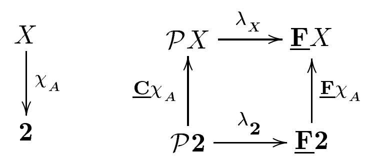
RESTE A FAIRE

3.2.3 Enumerating the transformations

NT004bc.txt The $4 \times 16 = 64$ natural transformations of Proposition 1 (page 8) are easily described.

Let $\underline{\mathbf{F}}$ be any contravariant functor $\underline{\mathbf{Ens}} \to \underline{\mathbf{Ens}}$.

Let λ be a natural transformation $\underline{\mathbf{C}} \to \underline{\mathbf{F}}$. This arbitrary transformation is totally determined by its value on $\{\mathbf{1}\}$ through the rule "*for each set X and each $A \subset X$, $\lambda_X\, A = \underline{\mathbf{F}}\chi_A\, \lambda_{\mathbf{2}}\{\mathbf{1}\}$*". Indeed, for any set X and any $A \in \underline{\mathbf{C}}X$, the naturality of λ entails the following commutative diagram

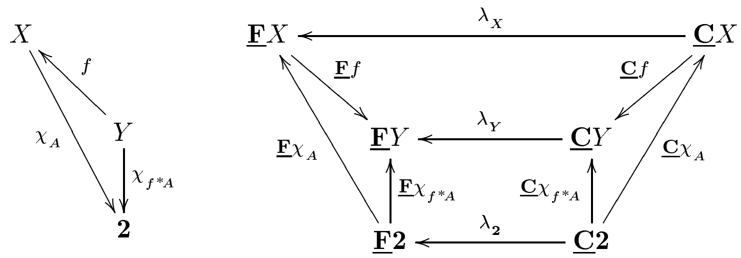


which reads $\lambda_X\, A = \lambda_X\, \underline{\mathbf{C}}\chi_A\{\mathbf{1}\} = \underline{\mathbf{F}}\chi_A\, \lambda_{\mathbf{2}}\{\mathbf{1}\}$.

Conversely, let us choose any $\Lambda \in \underline{\mathbf{F}}\mathbf{2}$. Then, the rule

$$\textit{for each set } X \textit{ and each } A \subset X,\ A \mapsto \underline{\mathbf{F}}\chi_A\, \Lambda \overset{def}{=} \lambda_X\, A \qquad (\diamondsuit)$$

defines a natural transformation $\lambda : \underline{\mathbf{C}} \to \underline{\mathbf{F}}$ satisfying $\lambda_{\mathbf{2}}\{\mathbf{1}\} = \Lambda$ Indeed, given $f : X \to Y$ we have:



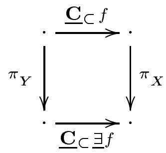
where the square with λ_Y and λ_X on its boundary is commutative because

$$\underline{\mathbf{F}}f\,\lambda_X A \overset{(\diamondsuit)}{=} \left(\underline{\mathbf{F}}f \circ \underline{\mathbf{F}}\chi_A\right)\Lambda = \underline{\mathbf{F}}(\chi_A \circ f)\,\Lambda = \underline{\mathbf{F}}\chi_{f^*A}\,\Lambda \overset{(\diamondsuit)}{=} \lambda_Y\, \underline{\mathbf{C}}fA;$$

Moreover, upon applying $(\diamondsuit)$ with $X = \mathbf{2}$ and $A = \{\mathbf{1}\}$, we have $\lambda_{\mathbf{2}}\{\mathbf{1}\} = \Lambda$; indeed, in this case $\chi_A = \chi_{\{\mathbf{1}\}} = \underline{\mathbf{Id}_2}$, which implies $\underline{\mathbf{F}}\chi_A\,\Lambda = \underline{\mathbf{Id}}_{\underline{\mathbf{F}}\mathbf{2}}\,\Lambda = \Lambda$; the result is then read *verbatim* from $(\diamondsuit)$.

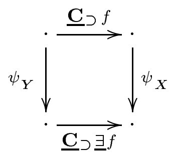
THE FRONT FACE:

$$\left\{\begin{array}{c} \underline{\textit{nothing (lack of adjoints)}} \\ \upsilon_Y\ \underline{\exists}_{\subset}\underline{\exists} f = \underline{\exists}_{\subset} f\ \upsilon_X \\ \underline{\mathbf{C}}_{\subset} f\ \upsilon_Y > \upsilon_X\ \underline{\mathbf{C}}_{\subset}\underline{\exists} f \quad \text{and} \quad \pi_Y\ \underline{\exists}_{\subset} f > \underline{\exists}_{\subset}\underline{\exists} f\ \pi_X \\ \pi_X\ \underline{\mathbf{C}}_{\subset} f = \underline{\mathbf{C}}_{\subset}\underline{\exists} f\ \pi_Y \quad \leftarrow \underline{\textit{the face}} \\ \underline{\forall}_{\subset}\underline{\exists} f\ \pi_X > \pi_Y\ \underline{\forall}_{\subset} \quad \text{and} \quad \underline{\textit{nothing (lack of adjoints)}} \end{array}\right\} \begin{array}{l} a3 \\ a2 \\ a1 \\ a0 \\ a1' \end{array}$$



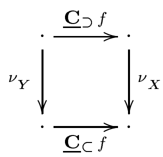
THE REAR FACE:

$$\left\{\begin{array}{c} \underline{\textit{nothing (lack of adjoints)}} \\ \delta_Y\ \underline{\forall}_{\supset}\underline{\exists} f = \underline{\forall}_{\supset} f\ \delta_X \\ \underline{\mathbf{C}}_{\supset} f\ \delta_Y > \delta_X\ \underline{\mathbf{C}}_{\supset}\underline{\exists} f \quad \text{and} \quad \psi_Y\ \underline{\forall}_{\supset} f > \underline{\forall}_{\supset}\underline{\exists} f\ \psi_X \\ \psi_X\ \underline{\mathbf{C}}_{\supset} f = \underline{\mathbf{C}}_{\supset}\underline{\exists} f\ \psi_Y \quad \leftarrow \underline{\textit{the face}} \\ \underline{\exists}_{\supset}\underline{\exists} f\ \psi_X > \psi_Y\ \underline{\exists}_{\supset} f \quad \text{and} \quad \underline{\textit{nothing (lack of adjoints)}} \end{array}\right\} \begin{array}{l} A3 \\ A2 \\ A1 \\ A0 \\ A1' \end{array}$$



THE TOP FACES (1):

$$\left\{\begin{array}{c} \underline{\exists}_{\subset} f\ \nu_X > \nu_Y\ \underline{\forall}_{\supset} f \quad \text{and} \quad \underline{\textit{nothing (lack of adjoints)}} \\ \nu_Y\ \underline{\exists}_{\subset} f = \underline{\forall}_{\supset} f\ \nu_X \\ \underline{\mathbf{C}}_{\supset} f\ \nu_Y > \nu_X\ \underline{\mathbf{C}}_{\subset} f \quad \text{and} \quad \nu_Y\ \underline{\forall}_{\supset} f > \underline{\exists}_{\subset} f\ \nu_X \\ \nu_X\ \underline{\mathbf{C}}_{\supset} f = \underline{\mathbf{C}}_{\subset} f\ \nu_Y \quad \leftarrow \underline{\textit{the face}} \\ \underline{\forall}_{\subset} f\ \nu_X > \nu_Y\ \underline{\exists}_{\supset} f \quad \text{and} \quad \nu_X\ \underline{\mathbf{C}}_{\subset} f > \underline{\mathbf{C}}_{\supset} f\ \nu_Y \\ \nu_Y\ \underline{\forall}_{\subset} f = \underline{\exists}_{\supset} f\ \nu_X \\ \underline{\textit{nothing (lack of adjoints)}} \quad \text{and} \quad \nu_Y\ \underline{\exists}_{\supset} f > \underline{\forall}_{\subset} f\ \nu_X \end{array}\right\} \begin{array}{l} c3 \\ c2 \\ c1 \\ c0 \\ c1' \\ c2' \\ c3' \end{array}$$



THE TOP FACES (2):

$$\left\{\begin{array}{c} \underline{\forall}_{\supset} f\ \nu_X > \nu_Y\ \underline{\exists}_{\subset} f \quad \text{and} \quad \underline{\textit{nothing (lack of adjoints)}} \\ \nu_Y\ \underline{\forall}_{\supset} f = \underline{\exists}_{\subset} f\ \nu_X \\ \underline{\mathbf{C}}_{\subset} f\ \nu_Y > \nu_X\ \underline{\mathbf{C}}_{\supset} f \quad \text{and} \quad \nu_Y\ \underline{\exists}_{\subset} f > \underline{\forall}_{\supset} f\ \nu_X \\ \nu_X\ \underline{\mathbf{C}}_{\subset} f = \underline{\mathbf{C}}_{\supset} f\ \nu_Y \quad \leftarrow \underline{\textit{the face}} \\ \underline{\exists}_{\supset} f\ \nu_X > \nu_Y\ \underline{\forall}_{\subset} f \quad \text{and} \quad \nu_X\ \underline{\mathbf{C}}_{\supset} f > \underline{\mathbf{C}}_{\subset} f\ \nu_Y \\ \nu_Y\ \underline{\exists}_{\supset} f = \underline{\forall}_{\subset} f\ \nu_X \\ \underline{\textit{nothing (lack of adjoints)}} \quad \text{and} \quad \nu_Y\ \underline{\forall}_{\subset} f > \underline{\exists}_{\supset} f\ \nu_X \end{array}\right\} \begin{array}{l} C3 \\ C2 \\ C1 \\ C0 \\ C1' \\ C2' \\ C3' \end{array}$$

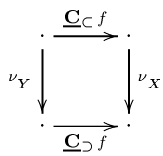
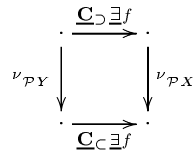


Figure 8: Expanding the π-ψ-cube (continued on next page)

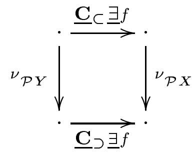
THE BOTTOM FACES (1):

$$\left\{\begin{array}{c}
\underline{\exists_{\subset}\exists} f\, \nu_{\mathcal{P}X} > \nu_{\mathcal{P}Y}\, \underline{\forall_{\supset}\exists} f \quad \text{and} \quad \underline{\textit{nothing (lack of adjoints)}} \\
\nu_{\mathcal{P}Y}\, \underline{\exists_{\subset}\exists} f = \underline{\forall_{\supset}\exists} f\, \nu_{\mathcal{P}X} \\
\underline{\mathbf{C}_{\supset}\exists} f\, \nu_{\mathcal{P}Y} > \nu_{\mathcal{P}X}\, \underline{\mathbf{C}_{\subset}\exists} f \quad \text{and} \quad \nu_{\mathcal{P}Y}\, \underline{\forall_{\supset}\exists} f > \underline{\exists_{\subset}\exists} f\, \nu_{\mathcal{P}X} \\
\nu_{\mathcal{P}X}\, \underline{\mathbf{C}_{\supset}\exists} f = \underline{\mathbf{C}_{\subset}\exists} f\, \nu_{\mathcal{P}Y} \quad \leftarrow \underline{\textit{the face}} \\
\underline{\forall_{\subset}\exists} f\, \nu_{\mathcal{P}X} > \nu_{\mathcal{P}Y}\, \underline{\exists_{\supset}\exists} f \quad \text{and} \quad \nu_{\mathcal{P}X}\, \underline{\mathbf{C}_{\subset}\exists} f > \underline{\mathbf{C}_{\supset}\exists} f\, \nu_{\mathcal{P}Y} \\
\nu_{\mathcal{P}Y}\, \underline{\forall_{\subset}\exists} f = \underline{\exists_{\supset}\exists} f\, \nu_{\mathcal{P}X} \\
\underline{\textit{nothing (lack of adjoints)}} \quad \text{and} \quad \nu_{\mathcal{P}Y}\, \underline{\exists_{\supset}\exists} f > \underline{\forall_{\subset}\exists} f\, \nu_{\mathcal{P}X}
\end{array}\right\}
\begin{array}{l} e3 \\ e2 \\ e1 \\ e0 \\ e1' \\ e2' \\ e3' \end{array}$$



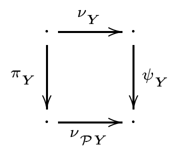
THE BOTTOM FACES (2):

$$\left\{\begin{array}{c}
\underline{\forall_{\supset}\exists} f\, \nu_{\mathcal{P}X} > \nu_{\mathcal{P}Y}\, \underline{\exists_{\subset}\exists} f \quad \text{and} \quad \underline{\textit{nothing (lack of adjoints)}} \\
\nu_{\mathcal{P}Y}\, \underline{\forall_{\supset}\exists} f = \underline{\exists_{\subset}\exists} f\, \nu_{\mathcal{P}X} \\
\underline{\mathbf{C}_{\subset}\exists} f\, \nu_{\mathcal{P}Y} > \nu_{\mathcal{P}X}\, \underline{\mathbf{C}_{\supset}\exists} f \quad \text{and} \quad \nu_{\mathcal{P}Y}\, \underline{\exists_{\subset}\exists} f > \underline{\forall_{\supset}\exists} f\, \nu_{\mathcal{P}X} \\
\nu_{\mathcal{P}X}\, \underline{\mathbf{C}_{\subset}\exists} f = \underline{\mathbf{C}_{\supset}\exists} f\, \nu_{\mathcal{P}Y} \quad \leftarrow \underline{\textit{the face}} \\
\underline{\exists_{\supset}\exists} f\, \nu_{\mathcal{P}X} > \nu_{\mathcal{P}Y}\, \underline{\forall_{\subset}\exists} f \quad \text{and} \quad \nu_{\mathcal{P}X}\, \underline{\mathbf{C}_{\supset}\exists} f > \underline{\mathbf{C}_{\subset}\exists} f\, \nu_{\mathcal{P}Y} \\
\nu_{\mathcal{P}Y}\, \underline{\exists_{\supset}\exists} f = \underline{\forall_{\subset}\exists} f\, \nu_{\mathcal{P}X} \\
\underline{\textit{nothing (lack of adjoints)}} \quad \text{and} \quad \nu_{\mathcal{P}Y}\, \underline{\forall_{\subset}\exists} f > \underline{\exists_{\supset}\exists} f\, \nu_{\mathcal{P}X}
\end{array}\right\}
\begin{array}{l} E3 \\ E2 \\ E1 \\ E0 \\ E1' \\ E2' \\ E3' \end{array}$$



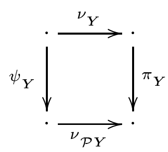
THE SIDE FACES (1):

$$\left\{\begin{array}{c}
\underline{\textit{nothing (lack of adjoints)}} \quad \text{and} \quad \delta_Y\, \nu_{\mathcal{P}Y} > \nu_Y\, \upsilon_Y \\
\upsilon_Y\, \nu_{\mathcal{P}Y} = \nu_Y\, \delta_Y \\
\nu_Y\, \upsilon_Y > \delta_Y\, \nu_{\mathcal{P}Y} \quad \text{and} \quad \pi_Y\, \nu_Y > \nu_{\mathcal{P}Y}\, \psi_Y \\
\psi_Y\, \nu_Y = \nu_{\mathcal{P}Y}\, \pi_Y \quad \leftarrow \underline{\textit{the face}} \\
\nu_{\mathcal{P}Y}\, \psi_Y > \pi_Y\, \nu_Y \quad \text{and} \quad \underline{\textit{nothing (lack of adjoints)}}
\end{array}\right\}
\begin{array}{l} b3 \\ b2 \\ b1 \\ b0 \\ b1' \end{array}$$



THE SIDE FACES (2):

$$\left\{\begin{array}{c}
\underline{\textit{nothing (lack of adjoints)}} \quad \text{and} \quad \upsilon_Y\, \nu_{\mathcal{P}Y} > \nu_Y\, \delta_Y \\
\delta_Y\, \nu_{\mathcal{P}Y} = \nu_Y\, \upsilon_Y \\
\nu_Y\, \delta_Y > \upsilon_Y\, \nu_{\mathcal{P}Y} \quad \text{and} \quad \psi_Y\, \nu_Y > \nu_{\mathcal{P}Y}\, \pi_Y \\
\pi_Y\, \nu_Y = \nu_{\mathcal{P}Y}\, \psi_Y \quad \leftarrow \underline{\textit{the face}} \\
\nu_{\mathcal{P}Y}\, \pi_Y > \psi_Y\, \nu_Y \quad \text{and} \quad \underline{\textit{nothing (lack of adjoints)}}
\end{array}\right\}
\begin{array}{l} B3 \\ B2 \\ B1 \\ B0 \\ B1' \end{array}$$



Therefore, listing the natural transformations from $\underline{\mathbf{C}}$ to a contravariant functor $\underline{\mathbf{F}}$ amounts to listing the elements Λ of $\underline{\mathbf{F}}\mathbf{2}$. In particular, if $\underline{\mathbf{F}}$ is any of the four functors $\underline{\mathbf{C}\,\exists}$, $\underline{\mathbf{C}\,\forall}$, $\underline{\exists\,\mathbf{C}}$, and $\underline{\forall\,\mathbf{C}}$ of Proposition 1 (page 8), we obtain the 16 natural transformations $\underline{\mathbf{C}} \xrightarrow{\lambda} \underline{\mathbf{F}}$ through formula ($\diamondsuit$) from the 16 elements Λ of $\mathcal{PP}\mathbf{2}$.

$\mathcal{PP}\mathbf{2}$ is naturally a boolean algebra (as a power set). Therefore, the 16 natural transformations $\underline{\mathbf{C}} \to \underline{\mathbf{F}}$, for any contravariant $\underline{\mathbf{F}} \in \{\underline{\mathbf{C}\,\exists}, \underline{\exists\,\mathbf{C}}, \underline{\mathbf{C}\,\forall}, \underline{\forall\,\mathbf{C}}\}$, as already observed at the end of section 1.2 (page 6), naturally form a boolean algebra for pointwise operations, like $\lambda \wedge \mu$ for $\lambda_{\mathbf{2}}\{\mathbf{1}\} \cap \mu_{\mathbf{2}}\{\mathbf{1}\}$, $\lambda \vee \mu$ for $\lambda_{\mathbf{2}}\{\mathbf{1}\} \cup \mu_{\mathbf{2}}\{\mathbf{1}\}$, *etc.* In order to use a notation that expresses this boolean algebra structure, we encode the elements od $\mathcal{PP}\mathbf{2}$ as follows: we first list the elements of $\mathcal{P}\mathbf{2}$ in the order given by the lexicographical order through the natural identification $\mathcal{P}\mathbf{2} = \mathbf{2}^{\mathbf{2}}$, that is $\emptyset, \{\mathbf{1}\}, \{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}$ (*i.e.* $(\mathbf{0}, \mathbf{0})$, $(\mathbf{0}, \mathbf{1})$, $(\mathbf{1}, \mathbf{0})$, $(\mathbf{1}, \mathbf{1})$). Then, we represent the subsets of $\mathcal{P}\mathbf{2}$ as binary arrays of length 4, a "1" in position j meaning that the j-th element of $\mathcal{P}\mathbf{2}$ is in Λ (*v.g.* 1010 is the code for $\{\emptyset, \{\mathbf{0}\}\}$, 0110 is the code for $\{\{\mathbf{1}\}, \{\mathbf{0}\}\}$, *etc.*). With this notation, the componentwise "and" and "or" correspond to $\cap$ and $\cup$ in $\mathcal{PP}\mathbf{2}$.

Given a contravariant functor $\underline{\mathbf{F}}$ with value $\mathcal{PP}X$ on X, we write λ^c the natural transformation corresponding to the choice of Λ represented by the "code" c. Tables 4, 5, 6 and 7 in the four following subsections contain, for $\underline{\mathbf{F}}$ respectiveley equal to $\underline{\mathbf{C}\,\exists}$, $\underline{\exists\,\mathbf{C}}$, $\underline{\mathbf{C}\,\forall}$ and $\underline{\forall\,\mathbf{C}}$, the 16 natural transformations $\underline{\mathbf{C}} \to \underline{\mathbf{F}}$; these tables characterize the set $\lambda^c_X A$, give some details of λ^c (*v.g.* it is π *etc.*), and give "the highest cw-property" of λ^c.

3.2.4 Case $C \to C\,\exists$

NT004bd.txt Let us consider first the case when $\underline{\mathbf{F}} = \underline{\mathbf{C}\,\exists}$. In this case, formula ($\diamondsuit$) (page 38) yields

$$\lambda^\Lambda_X A = (\underline{\mathbf{C}\,\exists})\chi_A\, \Lambda = \{B : \chi_A(B) \in \Lambda\}$$

Here is a description of all λ^Λ_X's:

- ▹ <u>The case when $c = 0000$</u> : Here $\Lambda = \emptyset$, *i.e.* $\lambda^{0000}_X A = \big\{B : \chi_A(B) \in \emptyset\big\} = \emptyset$. λ^{0000}_X is constant with value $\emptyset$; it is therefore cw-functorial for all combinations of spins, and by [AOS-3], when the target spin is $\underline{\mathbf{C}_{\supset}\exists}$, it satisfies cw-adjointness. By [AOS-1], the left-adjoint of $\lambda^{0000}_X : \underline{\mathbf{C}_{\subset}} \to \underline{\mathbf{C}_{\supset}\exists}$ is the constant map with value $\emptyset$, and the left-adjoint of $\lambda^{0000}{}_X : \underline{\mathbf{C}_{\supset}} \to \underline{\mathbf{C}_{\supset}\exists}$ is the constant map with value X.

- ▹ <u>The case when $c = 0001$</u> : Here $\Lambda = \big\{\{\mathbf{0}, \mathbf{1}\}\big\}$, *i.e.* $\lambda^{0001}_X A = \big\{B : \chi_A(B) \in \{\{\mathbf{0}, \mathbf{1}\}\}\big\}$. Thus $\lambda^{0001}_X A$ is the set of all $B \subset X$ meeting both A and $X \backslash A$. It is not monotonous, *i.e.* not componentwise functorial.

▷ <u>The case when $c = 0010$</u> : Here $\Lambda = \{\{\mathbf{0}\}\}$, *i.e.* $\lambda_X^{0010}A = \{B : \chi_A(B) \in \{\{\mathbf{0}\}\}\}$. Thus it is the set of all *non empty subsets* B of $X\backslash A$, *i.e.* $\lambda_X^{0100}\nu_X A$ (*see* below λ^{0100}, which corresponds to the "construct" of the set of all non-empty subsets). Therefore, the natural transformation $\lambda^{0010} : \underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\exists}$ is componentwise functorial when the spins given to the two occurrences of $\underline{\mathbf{C}}$ are opposite, but does not satisfy componentwise adjointness, just as λ^{0100}.

▷ <u>The case when $c = 0011$</u> : Here $\Lambda = \{\{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}\}$, *i.e.* $\lambda_X^{0011}A = \{B : \chi_A(B) \in \{\{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}\}\}$. Thus it is the set of all subsets B of X meeting $X\backslash A$. It follows that λ^{0011} is cw-functorial for pairs of opposite spins. The transformation λ^{0011} is the composite $\underline{\mathbf{C}}_{\subset} \xrightarrow{\nu} \underline{\mathbf{C}}_{\supset} \xrightarrow{\psi} \underline{\mathbf{C}}_{\supset}\underline{\exists}$; therefore it satisfies componentwise algebraicity (because ψ does), and the left-adjoint of λ_X^{0011} is $\nu_X\delta_X$ (*see* Figure 6). On the other hand, λ_X^{0011} has no right-adjoint for ψ has none (*see* 0101).

▷ <u>The case when $c = 0100$</u> : Here $\Lambda = \{\{\mathbf{1}\}\}$, *i.e.* $\lambda_X^{0100}A = \{B : \chi_A(B) \in \{\{\mathbf{1}\}\}\}$. Thus $\lambda_X^{0100}A$ is the set of all *non empty subset* of A, *i.e.* it is "nearly" π. The λ_X^{0100} are strictly increasing; λ^{0100} is componentwise functorial for pairs of identical spins. However, we do not have componentwise adjointness, no matter which spins we choose; indeed, λ_X^{0100} commutes neither with the empty intersection, because $\lambda_X^{0100}\big(\cap_{i\in\emptyset} A_i\big) = \mathcal{P}X - \{\emptyset\}$ and $\cap_{i\in\emptyset}\big(\lambda_X^{0100}A_i\big) = \mathcal{P}X$, nor with unions as is easily checked. One may also invoke [AOS-1].

▷ <u>The case when $c = 0101$</u> : Here $\Lambda = \{\{\mathbf{1}\}, \{\mathbf{0}, \mathbf{1}\}\}$, *i.e.* $\lambda_X^{0101}A = \{B : \chi_A(B) \in \{\{\mathbf{1}\}, \{\mathbf{0}, \mathbf{1}\}\}\}$. Thus $\lambda_X^{0101}A$ is the set of all subsets of X meeting A, *i.e.* it is ψ. It satisfies cw-functoriality for pairs of identical spins. As seen in Figure 6, $\psi_X : \underline{\mathbf{C}}_{\supset}X \to \underline{\mathbf{C}}_{\supset}\underline{\exists}\,X$ is ν-conjugate to $\pi_X : \underline{\mathbf{C}}_{\subset}X \to \underline{\mathbf{C}}_{\subset}\underline{\exists}\,X$, which is componentwise algebraic, since $\upsilon_X\pi_X = \mathbf{Id}_{\underline{\mathbf{C}}_{\subset}X}$ (*see* [AOS-2]). Therefore, ψ satisfies also componentwise algebraicity; its left-adjoint is δ_X. On the other hand, $\psi : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\supset}\underline{\exists}$ has no right adjoint for $\pi : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\subset}\underline{\exists}$ has no right-adjoint (*see* 1100).

▷ <u>The case when $c = 0110$</u> : Here $\Lambda = \{\{\mathbf{1}\}, \{\mathbf{0}\}\}$, *i.e.* $\lambda_X^{0110}A = \{B : \chi_A(B) \in \{\{\mathbf{1}\}, \{\mathbf{0}\}\}\}$. Thus $\lambda_X^{0110}A$ is the set of all nonempty subsets of X contained either in A or in $X\backslash A$. The components λ_X^{0110} are not monotonous, *i.e.* not functorial.

▷ <u>The case when $c = 0111$</u> : Here $\Lambda = \{\{\mathbf{1}\}, \{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}\}$, *i.e.* $\lambda_X^{0111}A = \{B : \chi_A(B) \in \{\{\mathbf{1}\}, \{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}\}\}$. Thus λ_X^{0111} is constant with value the set of all nonempty subsets of X; it is therefore cw-functorial (for all combinations of spins) without satisfying componentwise adjointness for any choice of spins (*see* [AOS-3]).

▷ The case when $c = 1000$: Here $\Lambda = \{\emptyset\}$, *i.e.* $\lambda_X^{1000} A = \{B : \chi_A(B) \in \{\emptyset\}\}$. Thus λ_X^{1000} is constant with value $\{\emptyset\}$; it is therefore cw-functorial for all combinations of spins, but does not satisfy cw-adjointness no matter which spin we choose (*see* [AOS-3]).

▷ The case when $c = 1001$: Here $\Lambda = \{\emptyset, \{\mathbf{0}, \mathbf{1}\}\}$, *i.e.* $\lambda_X^{1001} A = \{B : \chi_A(B) \in \{\emptyset, \{\mathbf{0}, \mathbf{1}\}\}\}$. Thus $\lambda_X^{1001} A$ is the set made of the empty set and the nonempty subset of X meeting both A ans $X \backslash A$. The components of λ^{1001} are not functorial.

▷ The case when $c = 1010$: Here $\Lambda = \{\emptyset, \{\mathbf{0}\}\}$, *i.e.* $\lambda_X^{1010} A = \{B : \chi_A(B) \in \{\emptyset, \{\mathbf{0}\}\}\}$. Thus $\lambda_X^{1010} A$ is the set of all subsets of $X \backslash A$; it satisfies cw-functoriality for pairs of opposite spins. λ^{1010} is the composite $\underline{\mathbf{C}}_{\supset} \xrightarrow{\nu} \underline{\mathbf{C}}_{\subset} \xrightarrow{\pi} \underline{\mathbf{C}}_{\subset}\underline{\exists}$; it satisfies componentwise algebraicity for π does. Its left-adjoint is $\nu_X \upsilon_X$. On the other hand, $\lambda^{1010} : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\supset}\underline{\exists}$ has no left-adjoints for $\pi : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\supset}\underline{\exists}$ has none.

▷ The case when $c = 1011$: Here $\Lambda = \{\emptyset, \{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}\}$, *i.e.* $\lambda_X^{1011} A = \{B : \chi_A(B) \in \{\emptyset, \{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}\}\}$. Thus $\lambda_X^{1011} A$ is the set made of the emptyset and of all subsets of X meeting $X \backslash A$. Therefore, $\lambda^{0010} : \underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\exists}$ is componentwise functorial when the spins given to the two occurrences of $\underline{\mathbf{C}}$ are opposite. It does not satisfy cw-adjointness for $\lambda^{1011}\nu = \lambda^{0100}$ which does not satisfy cw-adjointness.

▷ The case when $c = 1100$: Here $\Lambda = \{\emptyset, \{\mathbf{1}\}\}$, *i.e.* $\lambda_X^{1100} A = \{B : \chi_A(B) \in \{\emptyset, \{\mathbf{1}\}\}\}$. Thus $\lambda_X^{1100} A$ is the set of all subsets of A, *i.e.* it is π. It is cw-functorial for pairs if identical spins. As seen in Figure 6, $\pi : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\subset}\underline{\exists}$ satisfies cw-algebraicity, with left adjoint υ. On the other hand, $\pi : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\supset}\underline{\exists}$ has no left adjoints for it does not commute with unions (projective limits here).

▷ The case when $c = 1101$: Here $\Lambda = \{\emptyset, \{\mathbf{1}\}, \{\mathbf{0}, \mathbf{1}\}\}$, *i.e.* $\lambda_X^{1101} A = \{B : \chi_A(B) \in \{\emptyset, \{\mathbf{1}\}, \{\mathbf{0}, \mathbf{1}\}\}\}$. Thus $\lambda_X^{1101} A = \{\emptyset\} \bigcup \psi_X A$. λ_X is monotonous, *i.e.* λ^{1101} satisfies cw-functoriality for pairs of identical spins. However, it does not satisfy cw-adjointness no matter what the spins are for it does not commute with with intersections as is easily checked nor with unions for $\lambda_X \bigcup_{i \in \emptyset} A_i = \{\emptyset\}$ and $\bigcup_{i \in \emptyset} \lambda_X A_i = \emptyset$

▷ The case when $c = 1110$: Here $\Lambda = \{\emptyset, \{\mathbf{1}\}, \{\mathbf{0}\}\}$, *i.e.* $\lambda_X^{1110} A = \{B : \chi_A(B) \in \{\emptyset, \{\mathbf{1}\}, \{\mathbf{0}\}\}\}$. Thus $\lambda_X^{1110} A$ is the set made of all subsets of either A or $X \backslash A$. It is not monotonous, hence not cw-functorial no matter what the spins are.

▷ The case when $c = 1111$: Here $\Lambda = \{\emptyset, \{\mathbf{1}\}, \{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}\}$, *i.e.* $\lambda_X^{1111} A = \{B : \chi_A(B) \in \{\emptyset, \{\mathbf{1}\}, \{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}\}\}$. Thus $\lambda_X^{1111} A$ is constant with value $\mathcal{P}X$.

It satisfies cw-functoriality for all combinations of spins. From [AOS-3], it satisfies cw-adjointness when the target-spin is $\underline{\mathbf{C}_{\subset}\exists}$. By [AOS-1], the left adjoint for $\lambda_X^{1111} : \underline{\mathbf{C}}_{\subset}X \to \underline{\mathbf{C}}_{\subset}\exists X$ is the constant map with value $\emptyset$, and the left adjoint to $\lambda_X^{1111} : \underline{\mathbf{C}}_{\supset}X \to \underline{\mathbf{C}}_{\subset}\exists X$ is the constant map with value X.

For the sake of simplicity in writing Table 4, where we give all natural transformations $\underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\exists$, let us write $\mathfrak{v}$ for "$B = \emptyset$", $\mathfrak{m}$ for "$B \bigcap A \neq \emptyset$", and $\mathfrak{p}$ for "$B \subset A$" (respectively for "$\mathfrak{v}$ *oid*", "$\mathfrak{m}$ *eet*", "$\mathfrak{p}$ *art of*"), and let us use a *bar* as a negation symbol (*v.g.* $\overline{\mathfrak{m}}$). The elementary conditions $\chi_A B = u$ for $u = \emptyset, \{\mathbf{1}\}, \{\mathbf{0}\}, \{\mathbf{0}, \mathbf{1}\}$ are

condition 1	$\mathfrak{v}$	for $\chi_A B = \emptyset$ (*i.e.* $B = \emptyset$)
condition 2	$\overline{\mathfrak{v}}\,\mathfrak{p}$	for $\chi_A B = \{\mathbf{1}\}$ (*i.e.* $\emptyset \neq B \subset A$)
condition 3	$\overline{\mathfrak{v}}\,\overline{\mathfrak{m}}$	for $\chi_A B = \{\mathbf{0}\}$ (*i.e.* $\emptyset \neq B \subset X \backslash A$)
condition 4	$\overline{\mathfrak{v}}\,\mathfrak{m}\,\overline{\mathfrak{p}}$	for $\chi_A B = \{\mathbf{0}, \mathbf{1}\}$ (*i.e.* $A \bigcap B \neq \emptyset \neq B \bigcap X \backslash A$)

Each λ^c is characterized through a disjunction of some of these four conditions, condition k being present in the disjunction if and only if there is a 1 in position k of the binary writing of c (read from left to right). The two-page Table 4 is to be found on page 45.

3.2.5 Case $C \to \exists\, C$

NT004be.txt Let us consider next the case when $\underline{\mathbf{F}} = \underline{\exists\,\mathbf{C}}$. In this case, formula ($\Diamond$) (page 38) yields

$$\lambda_X A = (\underline{\exists \mathbf{C}})\chi_A \Lambda = \{\underline{\mathbf{C}}\chi_A u : u \in \Lambda\} = \big\{\{x \in X : \chi_A x \in u\} : u \in \Lambda\big\}$$

We have:

u	$\emptyset$	$\{\mathbf{1}\}$	$\{\mathbf{0}\}$	$\{\mathbf{0}, \mathbf{1}\}$
$\underline{\mathbf{C}}\chi_A u$	$\emptyset$	A	$X \backslash A$	X

.

It follows that, depending to Λ, $\lambda_X A$ is one of the 16 possible enumerations of subsets of X taken among $\emptyset$, A, $X \backslash A$, and X. Table 5 (page 47) describes the transformations $\underline{\mathbf{C}} \to \underline{\exists\,\mathbf{C}}$.

c	$B \in \lambda_X^c A$ if an only if	cw-functoriality
0000	False	for all combinations of spins
0001	$\overline{\mathfrak{v}}\,\mathfrak{m}\,\overline{\mathfrak{p}}$	
0010	$\overline{\mathfrak{v}}\,\mathfrak{m}$	for opposite spins
0011	$\overline{\mathfrak{v}}\,\mathfrak{m}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{m}\,\overline{\mathfrak{p}}$	for opposite spins
0100	$\overline{\mathfrak{v}}\,\mathfrak{p}$	for twice the same spin
0101	$\overline{\mathfrak{v}}\,\mathfrak{p}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{m}\,\overline{\mathfrak{p}}$	for twice the same spin
0110	$\overline{\mathfrak{v}}\,\mathfrak{p}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{m}$	
0111	$\overline{\mathfrak{v}}\,\mathfrak{p}$ **or** $\overline{\mathfrak{v}}\,\overline{\mathfrak{m}}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{m}\,\overline{\mathfrak{p}}$	for all combinations of spins
1000	$\mathfrak{v}$	for all combinations of spins
1001	$\mathfrak{v}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{m}\,\overline{\mathfrak{p}}$	
1010	$\mathfrak{v}$ **or** $\overline{\mathfrak{v}}\,\overline{\mathfrak{m}}$	for opposite spins
1011	$\mathfrak{v}$ **or** $\overline{\mathfrak{v}}\,\overline{\mathfrak{m}}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{m}\,\overline{\mathfrak{p}}$	for opposite spins
1100	$\mathfrak{v}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{p}$	for twice the same spin
1101	$\mathfrak{v}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{p}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{m}\,\overline{\mathfrak{p}}$	for twice the same spin
1110	$\mathfrak{v}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{p}$ **or** $\overline{\mathfrak{v}}\,\overline{\mathfrak{m}}$	
1111	$\mathfrak{v}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{p}$ **or** $\overline{\mathfrak{v}}\,\overline{\mathfrak{m}}$ **or** $\overline{\mathfrak{v}}\,\mathfrak{m}\,\overline{\mathfrak{p}}$	for all combinations of spins

Condition 4 : $\overline{\mathfrak{v}}\,\mathfrak{m}\,\overline{\mathfrak{p}}$, that is $\chi_A B = \{\mathbf{0}, \mathbf{1}\}$, that is $A \bigcap B \neq \emptyset \neq B \bigcap X \backslash A$

Condition 3 : $\overline{\mathfrak{v}}\,\overline{\mathfrak{m}}$, that is $\chi_A B = \{\mathbf{0}\}$, that is $\emptyset \neq B \subset X \backslash A$

Condition 2 : $\overline{\mathfrak{v}}\,\mathfrak{p}$, that is $\chi_A B = \{\mathbf{1}\}$, that is $\emptyset \neq B \subset A$

Condition 1 : $\mathfrak{v}$, that is $\chi_A B = \emptyset$, that is $B = \emptyset$

Table 4: (a) All natural transformations $\underline{\mathbf{C}} \to \underline{\mathbf{C}\exists}$

	$\diamond \xrightarrow{\lambda} \circ$	Description	The cw-left-adjoints.	Naturality?	
0000	$\mathfrak{e} : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\supset}\underline{\exists}$ $\mathfrak{e} : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\supset}\underline{\exists}$	$A \mathrel{+\!\!\!\longrightarrow} \emptyset$	$\mathfrak{e}_X \ : \underline{\mathbf{C}}_{\subset}X \leftarrow \underline{\mathbf{C}}_{\supset}\underline{\exists}X$ $\mathfrak{f}_X \ : \underline{\mathbf{C}}_{\supset}X \leftarrow \underline{\mathbf{C}}_{\supset}\underline{\exists}X$	$\emptyset \longleftarrow\!\!\!\!+ \mathbb{A}$ $X \longleftarrow\!\!\!\!+ \mathbb{A}$	yes yes
$\underline{0011}$	$\psi\,\nu : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\supset}\underline{\exists}$	$\psi\,\nu$	$(\nu\,\delta)_X \ : \underline{\mathbf{C}}_{\subset}X \leftarrow \underline{\mathbf{C}}_{\supset}\underline{\exists}X$	$(\nu\,\delta)_X$	yes
$\underline{0101}$	$\psi : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\supset}\underline{\exists}$	ψ	$\delta_X \ : \underline{\mathbf{C}}_{\subset}X \leftarrow \underline{\mathbf{C}}_{\supset}\underline{\exists}X$	δ_X	yes
$\underline{1010}$	$\pi\,\nu : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\subset}\underline{\exists}$	$\pi\,\nu$	$(\nu\,\upsilon)_X \ : \underline{\mathbf{C}}_{\supset}X \leftarrow \underline{\mathbf{C}}_{\subset}\underline{\exists}X$	$(\nu\,\upsilon)_X$	yes
$\underline{1100}$	$\pi : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\subset}\underline{\exists}$	π	$\upsilon_X \ : \underline{\mathbf{C}}_{\subset}X \leftarrow \underline{\mathbf{C}}_{\subset}\underline{\exists}X$	υ_X	yes
1111	$\mathfrak{f} : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\subset}\underline{\exists}$ $\mathfrak{f} : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\subset}\underline{\exists}$	$A \mathrel{+\!\!\!\longrightarrow} \mathcal{P}X$	$\mathfrak{e}_X \ : \underline{\mathbf{C}}_{\subset}X \leftarrow \underline{\mathbf{C}}_{\subset}\underline{\exists}X$ $\mathfrak{f}_X \ : \underline{\mathbf{C}}_{\supset}X \leftarrow \underline{\mathbf{C}}_{\subset}\underline{\exists}X$	$\emptyset \longleftarrow\!\!\!\!+ \mathbb{A}$ $X \longleftarrow\!\!\!\!+ \mathbb{A}$	yes yes

TABLE 4: (b) The natural transformations $\underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\exists}$ satisfying cw-adjointness and cw-algebraicity. The underlining of codings indicates the cw-algebraic transformations. In each case, the cw-adjoints are the components of a natural transformation.

c	$\lambda^c_X A$	cw-functoriality
0000	$\emptyset$	for all combinations of spins
0001	$\{X\}$	for all combinations of spins
0010	$\{X\backslash A\}$	
0011	$\{X\backslash A, X\}$	
0100	$\{A\}$	
0101	$\{A, X\}$	
0110	$\{A, X\backslash A\}$	
0111	$\{A, X\backslash A, X\}$	
1000	$\{\emptyset\}$	for all combinations of spins
1001	$\{\emptyset, X\}$	for all combinations of spins
1010	$\{\emptyset, X\backslash A\}$	
1011	$\{\emptyset, X\backslash A, X\}$	
1100	$\{\emptyset, A\}$	
1101	$\{\emptyset, A, X\}$	
1110	$\{\emptyset, A, X\backslash A\}$	
1111	$\{\emptyset, A, X\backslash A, X\}$	

(a) All natural transformations $\underline{\mathbf{C}} \to \underline{\exists}\,\underline{\mathbf{C}}$

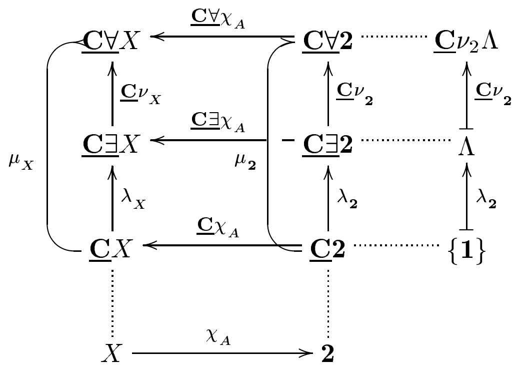
	$\diamond \xrightarrow{\lambda} \circ$	Values	The cw-left-adjoints. Naturality?		
0000	$\mathfrak{e} : \underline{\mathbf{C}}_{\subset} \to \underline{\exists}_{\supset}\underline{\mathbf{C}}$ $\mathfrak{f} : \underline{\mathbf{C}}_{\supset} \to \underline{\exists}_{\supset}\underline{\mathbf{C}}$	$A \mapsto \emptyset$	$\mathfrak{e}_X \ : \underline{\mathbf{C}}_{\subset} X \leftarrow \underline{\exists}_{\supset}\underline{\mathbf{C}} X$ $\mathfrak{f}_X \ : \underline{\mathbf{C}}_{\supset} X \leftarrow \underline{\exists}_{\supset}\underline{\mathbf{C}} X$	$\emptyset \leftarrow\!\!\!\mapsfrom \mathbb{A}$ $X \leftarrow\!\!\!\mapsfrom \mathbb{A}$	yes yes

(b) Natural transformations $\underline{\mathbf{C}} \to \underline{\exists}\,\underline{\mathbf{C}}$ satisfying cw-adjointness. There are only two transformations, which are cw-constant with value $\emptyset$. There are no cases of cw-algebraicity. In each case, the cw-adjoints are the components of a natural transformation.

Table 5: Natural transformations $\underline{\mathbf{C}} \to \underline{\exists}\,\underline{\mathbf{C}}$ with their properties

3.2.6 Case $C \to C\,\forall$

NT004bf.txt Let us consider the case $\underline{\mathbf{F}} = \underline{\mathbf{C}}\,\underline{\forall}$. We have the commutative diagram:



There is a bijection between natural transformations $\lambda : \underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\exists}$ and natural transformations $\mu : \underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\forall}$ given by the rule $\mu_X = \underline{\mathbf{C}}\nu_X\ \lambda_X$. Such a transformation μ is defined through the choice of some value $\Lambda \in \underline{\mathbf{C}}\,\underline{\forall}\mathbf{2} = \mathcal{P}\mathcal{P}\mathbf{2}$. If μ^Λ is the natural transformation $\underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\forall}$ corresponding to the choice of Λ, and λ^Λ the natural transformation $\underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\exists}$ corresponding also to the choice of Λ, we have

$$\mu_X^\Lambda = \underline{\mathbf{C}}\nu_X\ \lambda_X^{\underline{\mathbf{C}}\nu_{\mathbf{2}}\Lambda} \qquad (\alpha)$$

The analysis of the various λ^Λ given in section 3.2.3 translates directly into an analysis of the various μ^Λ. We observe that μ^Λ is cw-functorial if and only if $\lambda^{\underline{\mathbf{C}}\nu_{\mathbf{2}}\Lambda}$ is cw-functorial (because $\underline{\mathbf{C}}\nu_X$ is monotonous for all X), and that μ^Λ satisfies cw-adjointness if and only if $\lambda^{\underline{\mathbf{C}}\nu_{\mathbf{2}}\Lambda}$ satisfies cw-adjointness, in which case the left cw-adjoint $(\mu^\Lambda)^\sigma$ of μ^Λ is given for each X-component through $(\mu_X^\Lambda)^\sigma = (\lambda_X^{\underline{\mathbf{C}}\nu_{\mathbf{2}}\Lambda})^\sigma \underline{\mathbf{C}}\nu_X$ (with the same combination of spins as for $\lambda_X^{\underline{\mathbf{C}}\nu_{\mathbf{2}}\Lambda}$).

There is yet another "somehow obviously natural" construct $\mathcal{P}X \to \mathcal{P}\mathcal{P}X$ which we have not mentionned. It associates to each subset $A \subset X$ the set of *oversets* of A. We note it ω. It is μ^{0101} below. It is cw-functorial $\underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\forall}$ for pairs of opposite spins, and it is cw-algebraic for $\underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\subset}\underline{\forall}$.

One easily checks that if the code for Λ is *abcd*, then the code for $\underline{\mathbf{C}}\nu_{\mathbf{2}}\Lambda$ is *dcba*. From the description of the natural transformations $\underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\exists}$, we have therefore the following analysis.

- ▷ $\mu_X^{0000} A = \emptyset$;
- ▷ $\mu_X^{0001} A = \{X\}$;

▷ $\mu_X^{0010} A = \{B : X\backslash A \subset B \neq X\}$;

▷ $\mu_X^{0011} A = \{B : X\backslash A \subset B\}$; . . . . $\mu^{0011} = \omega\,\nu = \underline{\mathbf{C}}\nu\,\pi$;

▷ $\mu_X^{0100} A = \{B : A \subset B \neq X\}$;

▷ $\mu_X^{0101} A = \{B : A \subset B\}$; . . . . $\mu^{0101} = \omega = \underline{\mathbf{C}}\nu\,\pi\,\nu$;

▷ $\mu_X^{0110} A = \{B : (A \subset B \textbf{ or } X\backslash A \subset B) \textbf{ and } (B \neq X)\}$;

▷ $\mu_X^{0111} A = \{B : (A \subset B \textbf{ or } X\backslash A \subset B\}$;

▷ $\mu_X^{1000} A = \{B : X\backslash B \bigcap A \neq \emptyset \textbf{ and } X\backslash B \bigcap X\backslash A \neq \emptyset\}$;

▷ $\mu_X^{1001} A = \{B : X\backslash B \bigcap A \neq \emptyset \textbf{ and } X\backslash B \bigcap X\backslash A \neq \emptyset\} \bigcup \{X\} = \mu_X^{1000} A \bigcup \{X\}$;

▷ $\mu_X^{1010} A = \{B : X\backslash B \bigcap A \neq \emptyset\}$; . . . . $\mu^{1010} = \underline{\mathbf{C}}\nu\,\psi$;

▷ $\mu_X^{1011} A = \{B : X\backslash B \bigcap A \neq \emptyset\} \bigcup \{X\}$;

▷ $\mu_X^{1100} A = \{B : X\backslash B \bigcap X\backslash A \neq \emptyset\}$; . . . . $\mu^{1100} = \underline{\mathbf{C}}\nu\,\psi\,\nu$;

▷ $\mu_X^{1101} A = \{B : X\backslash B \bigcap X\backslash A \neq \emptyset\} \bigcup \{X\} = \mu_X^{1100} A \bigcup \{X\}$;

▷ $\mu_X^{1110} A = \{B : X \neq B \subset X\}$;

▷ $\mu_X^{1111} A = \mathcal{P}X$;

It results from relation (α) above that

$$\mu_X^{\Lambda} A = \{B : \chi_A(X\backslash B) \in \underline{\mathbf{C}}\nu_\mathbf{2}\Lambda\};$$

Now, $\underline{\mathbf{C}}\nu_\mathbf{2}\Lambda$ is a subset of $\big\{\emptyset, \{\mathbf{1}\}, \{\mathbf{0}\}, \{\mathbf{0},\mathbf{1}\}\big\}$, so that the elementary conditions whose disjunctions describe the belonging of B to $\mu_X^{\Lambda} A$ are

$$\chi_A(X\backslash B) = \emptyset \qquad \chi_A(X\backslash B) = \{\mathbf{1}\} \quad \chi_A(X\backslash B) = \{\mathbf{0}\} \quad \chi_A(X\backslash B) = \{\mathbf{0},\mathbf{1}\}$$

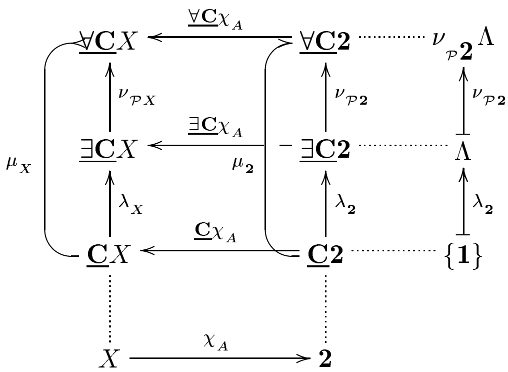
For the sake of simplicity in writing Table 6, where we give all natural transformations $\underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\forall}$, we write $\mathfrak{t}$ for "$B = X$", $\mathfrak{c}$ for "$B \bigcup A = X$", and $\mathfrak{l}$ for "B leaves something of A" (respectively for "$\mathfrak{t}$*otal*", "$\mathfrak{c}$*ompletes*" and "$\mathfrak{l}$*eaves*"), and we use a *bar* as a negation symbol (*v.g.* $\bar{\mathfrak{t}}$). The elementary conditions $\chi_A(X\backslash B) = \cdots$ given above are:

condition 1′	$\mathfrak{t}$	for $\chi_A(X\backslash B) = \emptyset$ (*i.e.* $B = X$)
condition 2′	$\mathfrak{c}\,\bar{\mathfrak{t}}$	for $\chi_A(X\backslash B) = \{\mathbf{1}\}$ (*i.e.* $X \neq B$ **and** $A \bigcup B = X$)
condition 3′	$\bar{\mathfrak{t}}\,\bar{\mathfrak{l}}$	for $\chi_A(X\backslash B) = \{\mathbf{0}\}$ (*i.e.* $A \subset B \neq X$)
condition 4′	$\bar{\mathfrak{t}}\,\mathfrak{l}\,\bar{\mathfrak{c}}$	for $\chi_A(X\backslash B) = \{\mathbf{0},\mathbf{1}\}$ (*i.e.* $A \bigcup B \neq X \neq B \bigcup X\backslash A$)

Each μ^c is characterized through a disjunction of some of these four conditions, condition k being present if and only if there is a 1 in position k of the binary writing of c *when reading it from the right to the left* (recall that if Λ has code *abcd*, then $\underline{\mathbf{C}}\nu_{\mathbf{2}}\Lambda$ has code *dbca*). The two-page table 6, to be found on page 51, gives all natural transformations $\mu : \underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\forall}$.

3.2.7 Case $C \to \forall\, C$

NT004bg.txt Let us consider the case $\underline{\mathbf{F}} = \underline{\forall}\,\underline{\mathbf{C}}$. We have the commutative diagram:



There is a bijection between natural transformations $\lambda : \underline{\mathbf{C}} \to \underline{\exists}\,\underline{\mathbf{C}}$ and natural transformations $\mu : \underline{\mathbf{C}} \to \underline{\forall}\,\underline{\mathbf{C}}$ given by the rule $\mu_X = \nu_{\mathcal{P}X}\, \lambda_X$. Such a transformation μ is defined through the choice of some value $\Lambda \in \underline{\forall}\,\underline{\mathbf{C}}\mathbf{2} = \mathcal{P}\mathcal{P}\mathbf{2}$. If μ^Λ is the natural transformation $\underline{\mathbf{C}} \to \underline{\forall}\,\underline{\mathbf{C}}$ corresponding to the choice of Λ, and λ^Λ the natural transformation $\underline{\mathbf{C}} \to \underline{\exists}\,\underline{\mathbf{C}}$ corresponding also to the choice of Λ, we have

$$\mu_X^\Lambda = \nu_{\mathcal{P}X}\, \lambda_X^{\nu_{\mathcal{P}\mathbf{2}}\Lambda} \qquad (\alpha')$$

The analysis of the various λ^Λ given in section 3.2.3 (page 44) translates directly into an analysis of the various μ^Λ. We observe that μ^Λ is cw-functorial if and only if $\lambda^{\nu_{\mathcal{P}\mathbf{2}}\Lambda}$ is cw-functorial (because $\nu_{\mathcal{P}X}$ is monotonous for all X), and that μ^Λ satisfies cw-adjointness if and only if $\lambda^{\nu_{\mathcal{P}\mathbf{2}}\Lambda}$ satisfies cw-adjointness, in which case the left cw-adjoint $(\mu^\Lambda)^\sigma$ of μ^Λ is given for each X-component through $(\mu_X^\Lambda)^\sigma = (\lambda_X^{\nu_{\mathcal{P}\mathbf{2}}\Lambda})^\sigma \nu_{\mathcal{P}X}$ (with the opposite source spin as in the case of $\lambda_X^{\underline{\mathbf{C}}\nu_{\mathbf{2}}\Lambda}$, and the same target spin).

One easily checks that if the code for Λ is *abcd*, then the code for $\nu_{\mathcal{P}\mathbf{2}}\Lambda$ is $((2^4-1)_{\mathbf{2}} - abcd)$ —the bitwise complement. From the description of the natural transformations $\underline{\mathbf{C}} \to \underline{\exists}\,\underline{\mathbf{C}}$, (α') above yields Table 7 (page 53).

c	$B \in \mu^{c}_{X} A$ if an only if	cw-functoriality
0000	False	for all combinations of spins
0001	$\mathfrak{t}$	for all combinations of spins
0010	$\mathfrak{c}\,\overline{\mathfrak{t}}$	for twice the same spin
0011	$\mathfrak{c}\,\overline{\mathfrak{t}}$ **or** $\mathfrak{t}$	for twice the same spin
0100	$\overline{\mathfrak{t}}\,\overline{\mathfrak{l}}$	for opposite spins
0101	$\overline{\mathfrak{t}}\,\overline{\mathfrak{l}}$ **or** $\mathfrak{t}$	for opposite spins
0110	$\overline{\mathfrak{t}}\,\overline{\mathfrak{l}}$ **or** $\mathfrak{c}\,\overline{\mathfrak{t}}$	
0111	$\overline{\mathfrak{t}}\,\overline{\mathfrak{l}}$ **or** $\mathfrak{c}\,\overline{\mathfrak{t}}$ **or** $\mathfrak{t}$	
1000	$\overline{\mathfrak{t}}\,\mathfrak{l}\,\overline{\mathfrak{c}}$	
1001	$\overline{\mathfrak{t}}\,\mathfrak{l}\,\overline{\mathfrak{c}}$ **or** $\mathfrak{t}$	
1010	$\overline{\mathfrak{t}}\,\mathfrak{l}\,\mathfrak{c}$ **or** $\mathfrak{c}\,\overline{\mathfrak{t}}$	for twice the same spin
1011	$\overline{\mathfrak{t}}\,\mathfrak{l}\,\overline{\mathfrak{c}}$ **or** $\overline{\mathfrak{t}}\,\overline{\mathfrak{l}}$ **or** $\mathfrak{c}\,\overline{\mathfrak{t}}$	for twice the same spin
1100	$\overline{\mathfrak{t}}\,\mathfrak{l}\,\overline{\mathfrak{c}}$ **or** $\overline{\mathfrak{t}}\,\overline{\mathfrak{l}}$	for opposite spins
1101	$\overline{\mathfrak{t}}\,\mathfrak{l}\,\overline{\mathfrak{c}}$ **or** $\overline{\mathfrak{t}}\,\overline{\mathfrak{l}}$ **or** $\mathfrak{t}$	for opposite spins
1110	$\overline{\mathfrak{t}}\,\mathfrak{l}\,\overline{\mathfrak{c}}$ **or** $\overline{\mathfrak{t}}\,\overline{\mathfrak{l}}$ **or** $\mathfrak{c}\,\overline{\mathfrak{t}}$	for all combinations of spins
1111	$\overline{\mathfrak{t}}\,\mathfrak{l}\,\overline{\mathfrak{c}}$ **or** $\overline{\mathfrak{t}}\,\overline{\mathfrak{l}}$ **or** $\mathfrak{c}\,\overline{\mathfrak{t}}$ **or** $\mathfrak{t}$	for all combinations of spins

Condition $1'$: $\mathfrak{t}$, that is $\chi_A(X \backslash B) = \emptyset$, that is $(B = X)$

Condition $2'$: $\mathfrak{c}\,\overline{\mathfrak{t}}$, that is $\chi_A(X \backslash B) = \{\mathbf{1}\}$, that is $(X \neq B$ **and** $A \bigcup B = X)$

Condition $3'$: $\overline{\mathfrak{t}}\,\overline{\mathfrak{l}}$, that is $\chi_A(X \backslash B) = \{\mathbf{0}\}$, that is $(A \subset B \neq X)$

Condition $4'$: $\overline{\mathfrak{t}}\,\mathfrak{l}\,\overline{\mathfrak{c}}$, that is $\chi_A(X \backslash B) = \{\mathbf{0}, \mathbf{1}\}$, that is $A \bigcup B \neq X \neq B \bigcup X \backslash A$

Table 6: (a) All natural transformations $\underline{\mathbf{C}} \to \underline{\mathbf{C}\forall}$

	$\diamond \xrightarrow{\mu} \circ$	Values	The cw-left-adjoints. Naturality?		
0000	$\mathfrak{e} : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\supset}\underline{\forall}$ $\mathfrak{e} : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\supset}\underline{\forall}$	$A \mapsto \emptyset$	$\mathfrak{e}_X : \underline{\mathbf{C}}_{\subset} X \leftarrow \underline{\mathbf{C}}_{\supset}\underline{\forall} X$ $\mathfrak{f}_X : \underline{\mathbf{C}}_{\supset} X \leftarrow \underline{\mathbf{C}}_{\supset}\underline{\forall} X$	$\emptyset \leftarrow\!\!\!+ \mathbb{A}$ $X \leftarrow\!\!\!+ \mathbb{A}$	yes yes
$\underline{0011}$	$\omega\,\nu : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\subset}\underline{\forall}$	$\underline{\mathbf{C}}\nu\,\pi = \omega\,\nu$	$\underline{\mathbf{C}}_{\subset} X \leftarrow \underline{\mathbf{C}}_{\subset}\underline{\forall} X$	$(\upsilon\,\underline{\mathbf{C}}\nu)_X$	yes
$\underline{0101}$	$\omega : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\subset}\underline{\forall}$	$\underline{\mathbf{C}}\nu\,\pi\,\nu = \omega$	$\underline{\mathbf{C}}_{\supset} X \leftarrow \underline{\mathbf{C}}_{\subset}\underline{\forall} X$	$(\nu\,\upsilon\,\underline{\mathbf{C}}\nu)_X$	yes
$\underline{1010}$	$\underline{\mathbf{C}}\nu\,\psi : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\supset}\underline{\forall}$	$\underline{\mathbf{C}}\nu\,\psi$	$\underline{\mathbf{C}}_{\supset} X \leftarrow \underline{\mathbf{C}}_{\supset}\underline{\forall} X$	$(\delta\,\underline{\mathbf{C}}\nu)_X$	yes
$\underline{1100}$	$\underline{\mathbf{C}}\nu\,\psi\,\nu : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\supset}\underline{\forall}$	$\underline{\mathbf{C}}\nu\,\psi\,\nu$	$\underline{\mathbf{C}}_{\subset} X \leftarrow \underline{\mathbf{C}}_{\supset}\underline{\forall} X$	$(\nu\,\delta\,\underline{\mathbf{C}}\nu)_X$	yes
1111	$\mathfrak{f} : \underline{\mathbf{C}}_{\subset} \to \underline{\mathbf{C}}_{\subset}\underline{\forall}$ $\mathfrak{f} : \underline{\mathbf{C}}_{\supset} \to \underline{\mathbf{C}}_{\subset}\underline{\forall}$	$A \mapsto \mathcal{P}X$	$\mathfrak{e}_X : \underline{\mathbf{C}}_{\subset} X \leftarrow \underline{\mathbf{C}}_{\subset}\underline{\forall} X$ $\mathfrak{f}_X : \underline{\mathbf{C}}_{\supset} X \leftarrow \underline{\mathbf{C}}_{\subset}\underline{\forall} X$	$\emptyset \leftarrow\!\!\!+ \mathbb{A}$ $X \leftarrow\!\!\!+ \mathbb{A}$	yes yes

TABLE 6: (b) The natural transformations $\underline{\mathbf{C}} \to \underline{\mathbf{C}}\,\underline{\forall}$ satisfying cw-adjointness and cw-algebraicity. The underlining of codings indicates the cw-algebraic transformations. In each case, the cw-adjoints are the components of a natural transformation.

c	$\lambda_X^{\mathfrak{c}} A$	cw-functoriality
0000	$\mathcal{P}X - \{\emptyset, A, X\backslash A, X\}$	
0001	$\mathcal{P}X - \{\emptyset, A, X\backslash A, \}$	
0010	$\mathcal{P}X - \{\emptyset, A, X\}$	
0011	$\mathcal{P}X - \{\emptyset, A\}$	
0100	$\mathcal{P}X - \{\emptyset, X\backslash A, X\}$	
0101	$\mathcal{P}X - \{\emptyset, X\backslash A\}$	
0110	$\mathcal{P}X - \{\emptyset, X\}$	for all combinations of spins
0111	$\mathcal{P}X - \{\emptyset\}$	for all combinations of spins
1000	$\mathcal{P}X - \{A, X\backslash A, X\}$	
1001	$\mathcal{P}X - \{A, X\backslash A\}$	
1010	$\mathcal{P}X - \{A, X\}$	
1011	$\mathcal{P}X - \{A\}$	
1100	$\mathcal{P}X - \{X\backslash A, X\}$	
1101	$\mathcal{P}X - \{X\backslash A\}$	
1110	$\mathcal{P}X - \{X\}$	for all combinations of spins
1111	$\mathcal{P}X$	for all combinations of spins

(a) All natural transformations $\underline{\mathbf{C}} \to \underline{\forall}\, \underline{\mathbf{C}}$

	$\diamond \xrightarrow{\mu} \circ$	Values	The cw-left-adjoints. Naturality?		
0000	$\mathfrak{f} : \underline{\mathbf{C}}_{\subset} \to \underline{\forall}_{\subset} \underline{\mathbf{C}}$ $\mathfrak{f} : \underline{\mathbf{C}}_{\supset} \to \underline{\forall}_{\subset} \underline{\mathbf{C}}$	$A \longmapsto \mathcal{P}X$	$\mathfrak{e}_X : \underline{\mathbf{C}}_{\subset} X \leftarrow \underline{\forall}_{\subset} \underline{\mathbf{C}} X$ $\mathfrak{f}_X : \underline{\mathbf{C}}_{\supset} X \leftarrow \underline{\forall}_{\subset} \underline{\mathbf{C}} X$	$\emptyset \longleftarrow\!\!\!+ \mathbb{A}$ $X \longleftarrow\!\!\!+ \mathbb{A}$	yes yes

(b) Natural transformations $\underline{\mathbf{C}} \to \underline{\forall}\, \underline{\mathbf{C}}$ satisfying cw-adjointness. There are only two, which are cw-constant with value $\emptyset$. There are no cases of cw-algebraicity. In each case, the cw-adjoints are the components of a natural transformation.

Table 7: Natural transformations $\underline{\mathbf{C}} \to \underline{\forall}\, \underline{\mathbf{C}}$ with their properties

4 Traveling from $\mathcal{P}^2$ to $\mathcal{P}$

5 Traveling from $\mathcal{P}^2$ to $\mathcal{P}^2$

References

[1] F.W., Lawvere, *Quantifiers and Scheaves*, Actes du congrès international des mathématiciens (1970), pp. 329-334.

[2] R. Guitart, *Relations et Carrés exacts*, Ann. Sc. Math Québec (1980), Vol. IV, No 2,pp. 103-125.