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Sense, Signs and Sketches in the Mathematical Invention of Coordination

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Abstract: In resolutions of problems and mathematical inventions, the notations (or signs) and the scopes (or senses) become intermingled in a kind of pulsation. For teaching it is worthwhile to preserve this pulsation and to work with it. Here at first, this pulsation is observed through the emergence of the idea of coordination in some cases (cartesian rectangular or oblique coordinates, tripolar coordinates, curvilinear coordinates). After that, for an hermeneutic *and* a semiotic analysis of this emergence of coordinations, we explain how significations and interpretations could be specified using arrows. Then coordinations could be understood as relational system of coordinates. And finally, thinking in terms of arrows and diagrams in the sense of the theory of categories, we emphasize that nowadays coordinations are nothing else but specifications of projective limits, equational structures, and sketches.

Keywords: notation, scope, pulsation, coordination, arrow, category, sketch.

1. Emergence of coordinations

Hereafter any formula (1) to (9) and any figure (1 to 7) is a sign, and in order to understand coordinations really, a good method would be to determine what signs are used, and, above all, how interactions are working between them.

1.1. Symptoms, characteristic equations, linear coordinates

In Euclid (1956, Proposition II, 14, p. 409) it is proved that given a circle C of diameter BF if we consider a point H on C and the perpendicular projection E of H on BF, then we get (fig. 1):

the square on HE is equal to the rectangle on BE, BF (1)

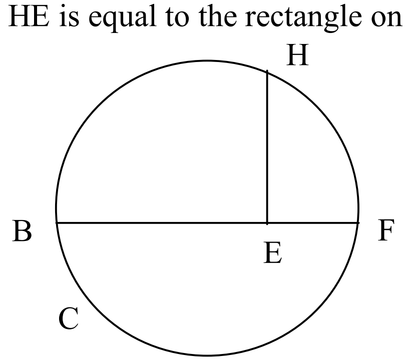

figure 1

This (1) is not an ''equation'' of the circle C, because it is neither a construction nor an assertion about any arbitrary point in the plane where the circle exists (this plane doesn't exist here as a mathematical object, the only real thing which is considered is the figure of the circle) ; the information given by fig.1 is a geometric relation which is defined and true on the circle. Furthermore (1) works for any circle, it is a *symptom* of any circle. Apollonius (1959) used this symptom for circles to produce symptoms for conics.

Nowadays, we metamorphose these symptoms into *characteristic equations*, namely (see e.g. Gramain, 1997, pp. 177-178):

$$y^2 = x(2p + px/a) \text{ (hyperbola)}, \quad y^2 = 2px \text{ (parabola)}, \quad y^2 = x(2p - px/a) \text{ (ellipse)}. \quad (2)$$

Those characteristic equations depend on an effective consideration of the plane, and in this plane of a convenient choice of coordinate axes. They could not be written in Descartes (1987), because in fact Descartes did not introduce really the so called ''cartesian axes''. He introduced the idea of arithmetization of geometry, by writing arithmetical relations among some lines in the figure. As a symptom, a ''cartesian equation'' is formed specifically on the curve ; but it is not a symptom, because now it is an arithmetical relation which is expressed. The discovery of Descartes is a method to provide such a cartesian equation for arbitrary curves.

The next step, the introduction of the plane and of axes in the plane, and so the formulation of characteristic equations, was achieved by Wallis (1655).

We could say that geometry and cartography where unified at this moment. The rectangular coordinates is a mapping, adapted to the basic shape of a square drawn in a plane as a reference (fig. 2.1), and at this step only became really a grid on the plane.

So, considering some given rectangular cartesian axes in a plane, it makes sense to ask for a geometrical characterisation of any second degree curves, and to prove (Wallis) that they are exactly conics. Euler proved that this fact is independent of the chosen rectangular axes. By modification of axes of coordinates, we can explain to what kind of conic corresponds an equation like:

$$ax^2 + 2bxy + cy^2 + ex + fy + g = 0. \quad (3)$$

According to Gino Loria, the first phase of the development of analytic geometry, is the period which begins with Descartes and Fermat and ends with Lagrange and Monge. It was at this time that the method of coordinates, outlined in the *Discours de la Méthode*, has finally become a body of doctrine providing for those who are studying geometry with formulas ready for use, applicable almost automatically, whatever may be the position of the coordinates axes. Loria thinks that an excellent but not yet perfect textbook on this matter is the Biot's book (Biot, 1834) ''Essai de géométrie analytique'' (first ed. 1802). In this book, in the setting of the rectangular grid in space, quadrics are systematically studied and classified.

After 1802, an important question still to be solved was the transformation of *oblique cartesian coordinates* into rectangular cartesian coordinates. The subject was studied (Loria, 1948) by Carnot, Livet, Français, Hachette, Lamé, Sturm, Cauchy. Lamé (1818) is interested particularly in this subject as a tool for the analysis of crystals. This is a very special case of what he will expand as the general theory of curvilinear coordinates (Lamé, 1859).

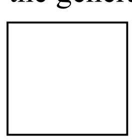

figure 2.1

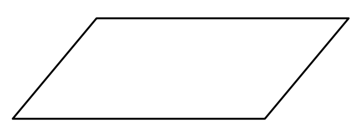

figure 2.2

The rectangular cartesian coordinates were adapted to the shape of a square, and now the oblique coordinates are adapted to the shape of a parallelogram (fig. 2.2). This produced a small but important freeing in the signification of equations. In an oblique cartesian system of axes, the previous equation (3) represents again an arbitrary conic, because the transformation from any oblique coordinates towards any rectangular coordinates is linear and bijective. But the specification of the type of conic which works in the rectangular case is no more valid directly.

A long time after, achieved only in the XXth century, we arrived at the general linear extension of rectilinear or oblique cartesian coordinates, the idea of coordinates in a vector space E with respect to a given basis. We shall need it in the continuation of our demonstration hereafter (see paragraph 4.2).

If a vector space E of finite dimension n is given, a basis in E could be described as the image of the canonical basis of R^n by a linear isomorphism

$$m : R^n \rightarrow E, \qquad (4.1)$$

and this isomorphism is of course determined by its inverse

$$p : E \rightarrow R^n, \qquad (4.2)$$

which itself is determined by its components $p_i : E \rightarrow R$, i = 1...., n.

Then the coordinates of an element x of E are $x_i = p_i(x)$, i = 1...., n, and so we write:

$$p(x) = (p_1(x), p_2(x),, \ldots, p_n(x)) = (x_1, x_2, \ldots, x_n) \quad (4.3)$$

1.2- Tripolar coordinates, from a symptom of the plane

In the XVIIth and XVIIIth centuries, polar, bipolar, spherical, cylindrical coordinates are used. Polar coordinates were used in Descartes's style, specifically on a given curve, by Cavalieri, Newton, James Bernoulli. The used of polar coordinates as a means of fixing any point in the plane and for a systematic study of any curve, is proposed by Jacob Hermann in 1729 (see Coolidge (1952), Boyer (1949, p. 76, Hermann (1729)). Newton (1736, p. 54 f) used bipolar coordinates, especially for the study of the `ellipses of the second order'' i.e. the ovals of Descartes. Newton observed (Boyer, 1949, p. 77) that Descartes handled ovals ''in a very prolix manner'', without the application of coordinates. If x and y are the distances of a variable point from two fixed poles, their relation for the ovals are

$$a+ex/d-y = 0 \quad (5)$$

Newton noted that if d = e, the curve becomes an equation for a conic section.

Lazare Carnot (1806, p.48) introduced the algebraic relation (a sum of 130 monomial terms) which is satisfied by the ten distances between five points A, B, C, D and E in the space. This formula is a law of the space, it expresses *the own coordination of the space*, when the space is endowed with its metric structure. In fact it is exactly for the space something like the symptom (1) for a circle, it is a metric symptom of the space. For A, B, C, D fixed, and the fixed quantities AD = f, AB = g, AC = h, BC = m, CD = n, BD = p, this symptom expresses, for any point E, the relation which is satisfied by the variable distances AE = l, BE = q, CE = r, DE = s. So E is located by such a 4-tuple (tetrapolar coordinates):

$$p(E) = t(E) = (l, q, r, s). \qquad (6)$$

These numbers are four and linked by one relation, and that is appropriated with the fact that the space is of dimension three ; they are the coordinates of E with respect to the ground which consists of the space endowed with its metric structure and with the four points A, B, C and D.

In 1841, in his first mathematical publication, Arthur Cayley (1841) expressed the Carnot's formulae as a determinant.

In the case of four points 1, 2, 3 and 4 in the plane, the formula of Cayley is (fig. 6) the following symbol [see also Blumenthal (1953, p. 99)]:

$$\begin{vmatrix} 0 & \underline{12}^2 & \underline{13}^2 & \underline{14}^2 & 1 \\ \underline{21}^2 & 0 & \underline{23}^2 & \underline{24}^2 & 1 \\ \underline{31}^2 & \underline{32}^2 & 0 & \underline{34}^2 & 1 \\ \underline{41}^2 & \underline{42}^2 & \underline{43}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0$$

figure 3

If we consider especially in the plane three distincts points 1, 2 and 3, with equal distances 12 = 21 = 23 = 32 = 31 = 13 = 1 (so 123 is an equilateral triangle), and if a point p is at distances p, q and r of 1, 2 and 3, p is located by (p, q, r) (tripolar coordinates) agreeing with the paraboloidal symptom

$$p^4 + q^4 + r^4 - (p^2q^2 + q^2r^2 + r^2p^2) - (p^2 + q^2 + r^2) + 1 = 0. \quad (7)$$

1.3- Curvilinear coordinates as families of surfaces or curves

The true general systematic approach of general curvilinear coordinates is due to Lamé, in the 1830’s (Guitart, 2009 a). The idea of curvilinear coordinates probably appeared in some Leibniz writing on coordination of systems of curves. The Lamé’s motivation was the study of physical problems as the question of the temperature of a given body, and he claimed that the more natural approach is to use of a system of coordinates adapted to this body, with family of level surfaces parallel or orthogonal to the body. The analytical signs of curvilinear coordinates is by a formula of transformations from rectangular cartesian coordinates (see (8.1) and (8.2)).

For example, in the two dimensional case, if the body is an ellipse, then rather than rectangular or polar coordinates it would be better to use of an elliptical system of coordinates (fig. 4.1 and fig. 4.2), constituted of confocal ellipses and hyperbolas.

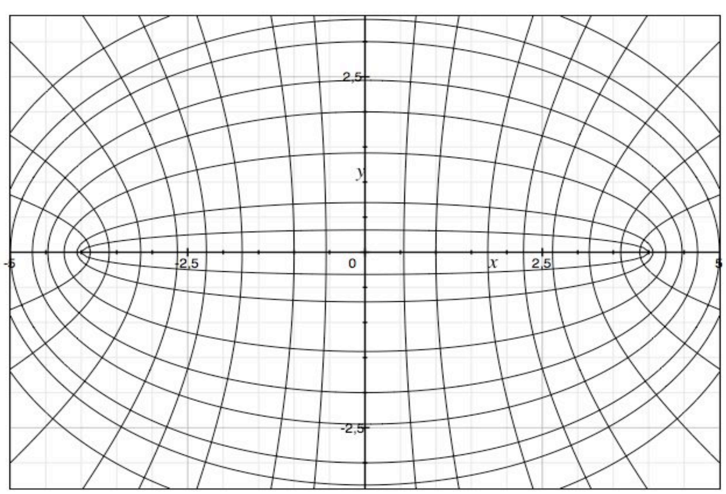

figure 4.1.

$$\frac{x^2}{a^2\cosh^2\mu} + \frac{y^2}{a^2\sinh^2\mu} = 1$$

$$\frac{x^2}{a^2\cos^2\nu} - \frac{y^2}{a^2\sin^2\nu} = 1$$

figure 4.2

A point P of cartesian coordinates (x, y) is located by (μ, ν) such that P belongs to the two corresponding curves. The transformation from (μ, ν) to (x, y) is given by

$$m(\mu,\nu) = (x, y), \text{ with } x = a \cosh \mu \cos \nu, \quad y = a \sinh \mu \sin \nu, \qquad (8.1)$$

and conversely the elliptical coordinates are given by

$$p(x,y) = e((x,y)) = (\mu, \nu). \qquad (8.2).$$

2. Arrows

Firstly we argue about the fact that any sign ''is'' an arrow. The games of these arrows provide a coordination of the semiosis, and this is the structural part of the sense. Secondly we precise that in the case of mathematical discourses, the signs work as diagrams and as abbreviations.

The system of signs exhibited to introduce coordinations hereover, could be understood, in a very general acceptation, in terms of arrows.

2.1- Semiotics and hermeneutics, structure and culture

A basic presentation of the problematics of signs and semiotics is given by Eco (1988) ; and a clear synthetic presentation of hermeneutics is given by Grondin (2006). We would like to stress upon the fact that the full sense of a discourse has two complementary and interacting components, semiotic and hermeneutic.

On the one hand (semiotics) any discourse is inscribed in a language, as a phonological production of signs, and there it has a dynamical *structure*.

We know with Jakobson (1976, p. 78) that a phoneme is a sheer differential sign: the only semiotic content of a phoneme is its difference from others phonemes. So the structure consists of variations and dispositions of phonemes, and it is understandable through coding process and grammatical analysis.

Ultimately, the question of meaning or signification of a discourse, is the problem of articulation and dynamical functioning of a system of *representations* by signs (linguistic signs). A minimal definition of a sign is in Eco (1988, p. 45) as an entity which could have a signified object.

In the Peirce view, this signified or object is the target of an arrow, the body of the arrow itself is the interpretant, and the source of the arrow is the signifier or signifying element. We represent that in (fig. 5.1). In the perspective of Savan (1988) or Lizka (1996), we have to treat interpretants as translations, or ''translatants''. Furthermore, each of the three components of a sign could be a sign again. The sense (seen as a global holding of a complex of meanings) is nothing else than the shape of the system, the shape of the structure, as a result of combinations of arrows of signs.

On the other hand (hermeneutics), any discourse is dedicated to a promotion of values, and so participates to the open construction of a social *culture*.

The sense is a question of hermeneutics, that is to say a question of *interpretation*, in the rhetorical tradition; more accurately it is a question of value in the free commerce of interpretations. In principle an interpretation pretends to provide an understanding of what truly the discourse would like to say. So it is a question of translation and elucidation of something which is obscure or at least incomplete. Speaking with the words of Droysen, Dilthey stressed on this point: the goal of hermeneutics is to construct a historical comprehension (Verstehen) rather than a scientifical explanation (Erklären) ; comprehension is delivered as another discourse which draws a kind of arrow in the culture, which is the indication of an orientation from a given point of view. Furthermore, in the Gadamer's view, the interpretation depends of a genuine implication of the interpreter, it is a performance, the living gesture of a human being.

So, for us, the hermeneutical sense is a kind of arrow (fig. 5.2), the body of which is the subjective act of interpretation, with target the comprehension in discourse given by the interpretation in the realm of « parole », and the source being the initial discourse which was to be interpreted.

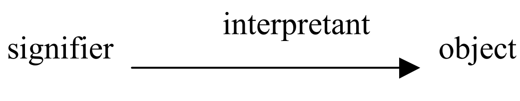

figure 5.1

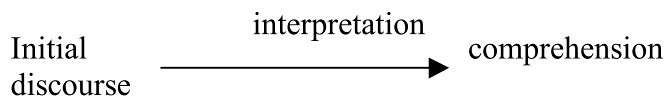

figure 5.2

On the semiotic side the sense (or meaning) resides in a methodical new combining of some existing arrows, and on the hermeneutic side the sense resides in the subjective elaboration of a single new arrow.

Of course we are free to envisage a given culture as a system of cultural signs (including generally some linguistical signs), and so the semiotics acts at this level of hermeneutics as well as at the linguistical level. Conversely, a sign considered as an arrow according to the perspective of Savan and Liszka, is a kind of elementary interpretation, an element in the hermeneutic realm.

For a given discourse, we have two stakes: its meaning as a shape of an articulation of signifying representations, its sense as a comprehensive interpretation in a culture. Furthermore, the discourse holds these two stakes in a living interaction, through the *gesture* of speaking (parole). Someone does the act of speaking the discourse, addressing the structure (of a system of phonemes) to someone who is living in a given culture.

The question of the full sense of a discourse now is the question of how in the process of « parole », the structure and the culture interact all together, around the given discourse. So the emergence of interpretation is in the realm of semiosis, as well structural semiosis is not separable from games of interpretations. We propose to take care of this observation by thinking in terms of diagrams of arrows (like those in fig. 5.1 and fig. 5.2).

2.2- The case of a mathematical discourse

Mutatis mutandis, we can transpose or particularize the previous remark (paragraph 2.1), valid for any discourse, to the special case of a mathematical discourse. There the act of ''speaking'' is replaced by the act of ''doing mathematics'', inside the mathematical language. The production of signs is done by mathematical scriptures of computations and figurations. The full sense is performed by confrontation of the given structure of these scriptures (seen also as a system of *mathematical representations*) with the mathematical history and culture (old theorem, theories and problems) and this provides a *mathematical interpretation*.

From a semiotical point of view we have in this mathematical game two intertwined levels: at a ground level we have signs for direct scriptural mathematical representations, and at a higher level we could use of signs for large and complex mathematical interpretations. In this conception, any mathematical data could be understood as an arrow (fig. 6.1) which could be read: ''from the point of view f, the object A stands for the object B'', as well as: ''f is a difference between B and A''.

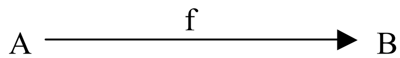

figure 6.1

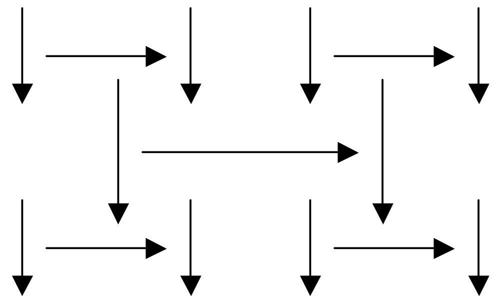

figure 6.2

This equivocation on the sense of an arrow (it could mean an identification or a distinction) is in fact essential to the living use of arrows in mathematics. We call that *the initial pulsation of the arrow*. In the saggital world the basic problem (and tool) is to create arrows, by serial or parallel compositions. An icon of our organizing thoughts is (fig. 6.2).

2.3- Coordinations, diagrams, abbreviations

In this paragraph we would like to make clear why mathematics is an art of inventions of necessities by modifications of diagrams. This will be used at the beginning of paragraph 4.1.

Mathematics neither is the science of number and space, nor is the science of logic or physics, although these subject-matters are those to which mathematics has been extensively applied.

A better thesis by Cassius Keyser (1933) is that mathematical thinking is postulational thinking. For Keyser every successful adventure in postulational thinking eventuates in the establishment of a hypothetical doctrinal function; the function is composed of propositions each of which asserts that some propositional form is logically implied by a set of other such forms. So, for Keyser, the mathematical questions are those about the world of the logically possible (whereas the scientific questions are those about the Actual world). The coordinate enterprises — mathematics and science — together embrace the whole knowledge-seeking activity of man (by ''knowledge'' is meant such knowledge as is expressible by propositions). Their combined scope is the two-fold world of the actual and the logical possible — the world of propositional fact and the world of propositional form, the world whose truth is discoverable by none but empirical thinking, wherein observation is sovereign, and the world whose truth is discoverable by none but postulational thinking, where deduction is sovereign. So, for Keyser, the validity of mathematical propositions is independent of the actual world — the world of existing subject-matters is logically prior to it, and would remain unaffected were it to vanish from being.

Nevertheless this thesis is too much tied to the logical point of view on mathematics, and we would rather prefer to think in terms of coordinations than in terms of logic. Our claim is: *Mathematics is the art of invention of necessary coordinations in the world of possible.*

Coordinations are nothing else that synthetico-analytical methods for any special science, e.g. for logic, arithmetic, probability, geometry, physics, etc. Conversely, anything in a given specific science which is a pure fact of necessary coordination, is a true mathematical point. So in the history of mathematics, the mathematical coordinations have been discovered in some scientific contexts, e.g. in arithmetic, in geometry, in logic, in physics, etc.

Arithmetic and geometry are sciences derived from the core of mathematical thinking, when the mathematician constructs an interpretation of his work in terms of hearing or sight. Arithmetic and geometry as sciences are already on the side of meaning and sense of mathematics: historically they are two basic ways of interpretation of mathematics. This point applies also to logic and physics.

As well as sciences are domains of applications for mathematics, they are sources of new mathematical ideas, as far as they stimulate new practices about coordinations.

Let us observe that, with respect to the Keyser distinction, a true working mathematician is in fact simultaneously a pure mathematician and a scientist, because on working and computing and deducing he is also observing the mathematical things, and he discovers through these observations. For him the world of mathematical things is the actual world, his observations are assumed as feelings with internal senses inside comprehension. In fact these observations are also actions of a mathematical value, as far as they generate new rigourous coordinations.

In a way the conception of Keyser is near from the conception of Charles Saunders Peirce which defines mathematics as ''the study of hypothetical states of things'', and, according to a wording in 1870 of his father Benjamin Peirce: ''the science which draws necessary conclusions'' (Peirce, 1967, 227-244).

However the Keyser's description is perhap too much logicist and it does not encounter enough the question of poietic and invention of mathematics. In a recent paper Daniel G. Campos (2007) examined Peirce's propositions in this direction, and he wrote that Peirce's position is that the creation of mathematical hypotheses is poietic, but is not merely poietic, and accordingly, that hypothesis-framing is part of mathematical reasoning that involves an element of poiesis but is not poietic either. So Campos proposed that hypothesis-making in mathematics stands between artistic and scientific poietic creativity with respect to imaginative freedom from logical and actual constraints upon reasoning.

Peirce treated sign theory as central to his work on logic, as the medium for inquiry and the process of scientific discovery. So in Peirce perspective, everything in the world is made of signs, of living combinations and productions of signs (the phenomenon of semiosis).

In analysing the mathematical thinking and productions, it will be crucial to exhibit the mechanism of invention of coordinations in the perspective of the semiosis, and to understand construction of coordination through semiotic elaborations. In Peirce's terminology a diagram is a special sign, an icon in fact, exhibiting existing relations among parts of a state of things (Chauviré, 2008 p. 36) ; and in Peirce's view, *the basic mathematical action is precisely the construction and modification of diagrams*.

There are diagrams with an arithmetical flavor (equational formulas) or with a geometrical tendency (geometrical figures), and the « dialectic » between these two aspects is of great importance in the development of mathematics ; in some sense the various mathematical results on this point (e.g. principles of duality) express a special mathematical *pulsation* (Guitart, 1999, 2008) in the mind of the thinker when he has to choose a sense for interpretation to direct the meaning of his thinking. An example is the famous Cayley's diagrams for groups (as well as the diagrammatical forms of Kempe (1886), as a vehicle for the pulsation between the algebraic or arithmetical (tabular) description of a group and its geometrical counterpart.

This pulsation between arithmetic and geometry indicates a moment of an inventive gesture in the realm of mathematics. The pulsation is sollicited by a diagram that we have to modify, and this is possible because a diagram always bears in itself an *abbreviation* (Guitart, 2000, p. 162), i.e. an arrow which comes from a ground and is going to a functionality (fig. 7) which provides an orientation in the mathematical knowledge.

$$\text{old ground} \xrightarrow{\text{abbreviation}} \text{new functionality}$$

figure 7

To this hermeneutic fact corresponds at the sheer semiotical level the fact that a diagram is always incomplete and so is open.

We speak of *pulsation* and *abbreviation* ; other autors as Châtelet G. (1993) and Alunni (2004), speak of a virtual dimension of any diagram seen as an abstract-machine, prior to any representation. We agree that a diagram is a static picture inciting anyone to a *gesture* of modification of this given diagram. The mini-model of that is precisely the *initial pulsation of the arrow* (fig. 6.1).

2.4- Maps as arrows abridging a system of arrows

Today everywhere in mathematics we employ maps or functions

$$f : E \to F : x \mapsto f(x) = y \qquad (9)$$

from a set E to a set F, such a map being the datum of a set E, a set F, and the attribution to any x in E of an image f(x) in F.

We have to stress on the following point: stricto sensus a map is a glueing of flow of arrows (fig. 13) ; so the map $f : E \to F$ is an arrow which is not so ‘’concrete’’, it is an abstract abbreviation for an abstract system of arrows. The idea of the pulsation of the arrow (fig. 6.1) is made more dynamic when we consider maps (fig. 8), which can be seen as a flow in the semiosis (fig. 6.2).

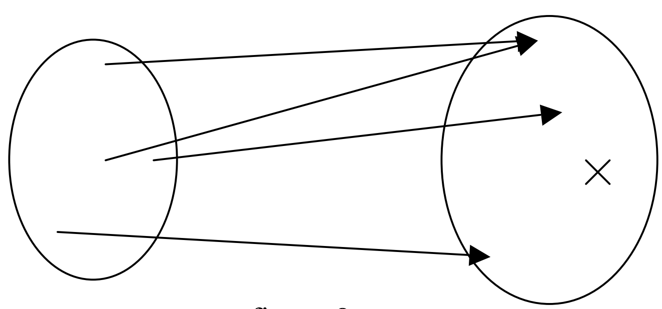

figure 8

On the one hand, by an analysis following 2.1, 2.3 and 2.3., a system of curves as in the picture (fig. 4) could be understood in the general semiotical framework in terms of sytems of signs or arrows. But on the other hand, now we can see it as an indication for a construction of functions and maps. In the history of mathematics, the developpment toward the final picture (fig. 8) for a function is related to inventions of curves and coordinations. So, in the modern setting, coordinations are constructions of maps, i.e. of arrows in the category of sets. At the mathematical level the general coordination of the semiosis is expressible in terms of maps, or even in terms of arrows abstractly organized in categories and diagrams.

3. Relational coordinations

In 1953-54 in the vein of Peirce's relational logic, in the framework of the calculus of binary relations, an abstract analysis of coordination was introduced by Riguet (1953, 1954) under the name of a ''relational system of coordinates''. Let R_i be the equivalence relation (congruence) generated on E by the projection p_i (see (4.3)) i.e.

$$xR_iy \quad \Leftrightarrow \quad p_i(x) = p_i(y) \qquad (10)$$

We can forget the external data p_i and just consider the set E as equipped with the family (R_i) of binary relations, and so $(E, (R_i))$ is a relational structure, with:

$$R\hat{}_i = \cap_{j\neq i} R_j, \quad R_i \cap R\hat{}_i = \Delta_E, \quad R_i R\hat{}_i = E\times E. \qquad (11)$$

Clearly any curvilinear coordinates could be seen as an example of a relational coordination. But this notion allows also the consideration of decomposition of an algebraic structure as a cartesian product of other structures of the same type (Chatelet, A., 1954, 1956,1966).

In fact, a map is a special case of a binary relation between two sets, and the calculus of composition and reversions of maps is a part of the general *calculus of relatives* introduced by Peirce. Nowadays, this calculus is understood as the work in the category of binary relations. So the Riguet's presentation is worthwile by itself, but also could be interpreted in the framework of the involutive category of correspondances.

4- Sketches

We conclude with the consideration of diagrams in the technical sense in category theory. Of course these diagrams are also diagrams according to Peirce, but some very special ones.

4.1- Coordinations as categorical diagrams

Taking seriously into account these ideas (paragraph 2.3) that mathematics is the art of invention of necessary coordinations in the world of possible, and the basic mathematical action is precisely the construction and modification of diagrams, now we claim that *we need only categorical diagrams in order to describe the fonctionality of the mathematical thinking*.

The main fundamental elementary tool in Category Theory is the Yoneda lemma. We can understand it (Guitart, 2007)) as a principle allowing to forget how the objects and arrows had been constructed, and to work only with the fact of relative interactions between objects; we speak of a scooping-out of objects and the consideration of the outside as the true substance of which objects are made of. So ultimately any object X in any category C ''is'' a system of arrows, namely the category C/X having as objects the arrows of target X in C, and this category C/X is thought as the natural shape of X in C. So, any object X is located by the shape of the system of links from others to itself; this is a kind of coordinates systems of X tied up with its background C.

More specifically we have to show how mathematical coordinations could be realized in the framework of the saggital icon (fig. 6.2) or more accurately in terms of maps, and even as a construction of a categorical diagram. This will be true for any mathematical coordination, but here now we will restrict our explanations to the case of geometrical coordinations seen hereover. Keeping in mind the general abstract setting on arrows initiated in paragraph 2, we strengthen the part play by arrows in coordinations, to extract the operational categorical content from the various diagrams seen hereover in the paragraph 1 (symptoms, equations, graphics). In this way coordination will be directly related to categorical diagrams and the modern idea of a sketch. In this process we emphasize the fonctionality of coordinates, loosing its initial signs. For instance the initial signs for elliptical coordinates (fig. 4.1-2) are replaced by a sketch (fig. 14).

4.2- Projective and mixed sketches

In the case of the tripolar coordinates associated to a symptom of the plane given by f (see paragraph 1.2) a point of the plane is represented by a surabundant number of coordinates, but these are submitted to a constraint ($f(p, q, r) = 0$). So the plane is represented as a kernel k of the map f, and this is a projective limit.

In fact this observation works in the general context of curvilinear coordinates, as follows.

The linear isomorphism $m : R^n \to E$ (paragraph 1.1 on linear coordinates) is as a geographical map which allows us to locate points x in E by using coordinates $p(x) = X$ (or $m(X) = x$). It is a coordination of E, and with this data E is more structured ; so we think also of this data as a structuration of E by a decomposition law.

In fact now, for curvilinear coordinates (paragraph 1.3), *we can eliminate the condition that* m (*or* p) *is linear* in (4.2); it is sufficient to consider an abstract set E and a bijection $m : R^n \to E$, or its inverse bijection $p : E \to R^n$ (e.g. the map e in (6)) or even just a map not necessarily bijective

$$p : E \to R^n \qquad (12)$$

So for the elliptical coordinates we get $p(M) = e(x, y) = (\mu, \nu)$.

By composition of p with the canonical projectios $pr_i : R^n \to R$, $i = 1...., n$, the datum of p is equivalent to the system of maps $p_i = pr_i.p$, $i = 1...., n$, i.e. a cone-shaped diagram (fig. 14) with top E related to the ''projective limit'' cone with top R^n through a unique arrow $p : E \to R^n$ factorizing the cone (p_i) through (pr_i) :

$$(p_i) = (pr_i).p. \quad (13)$$

If the emphasis is kept on the map $m : R^n \to E$, we get a dual approach to curvilinear coordinates, as parametrization, which could be generalize to a datum $m : U \to E$, with an open subset U included in R^n ; if we extend that to the idea of a manifold as a datum of a coherent family of such local parametrizations (m_j, U_j), it is capturable in the notion of an inductive limit and a mixed sketch (fig. 9), with a factorization:

$$(s_j) = p(m_j) \quad (14)$$

The true complete sketch is more complicated. After the first projective specification, we have to continue, to precise in details what each projection p_i is made of, etc. In order to do that we have to add material: basic maps (as cos x, ch x,…), compositions of such maps, equations among these compositions, and so on.

The datum

$$(p_{i,j}) = (pr_i.p.m_j) \qquad (15)$$

is a *matrix* of p, its analytical coordination in the complete sketch associated to fig. 9.

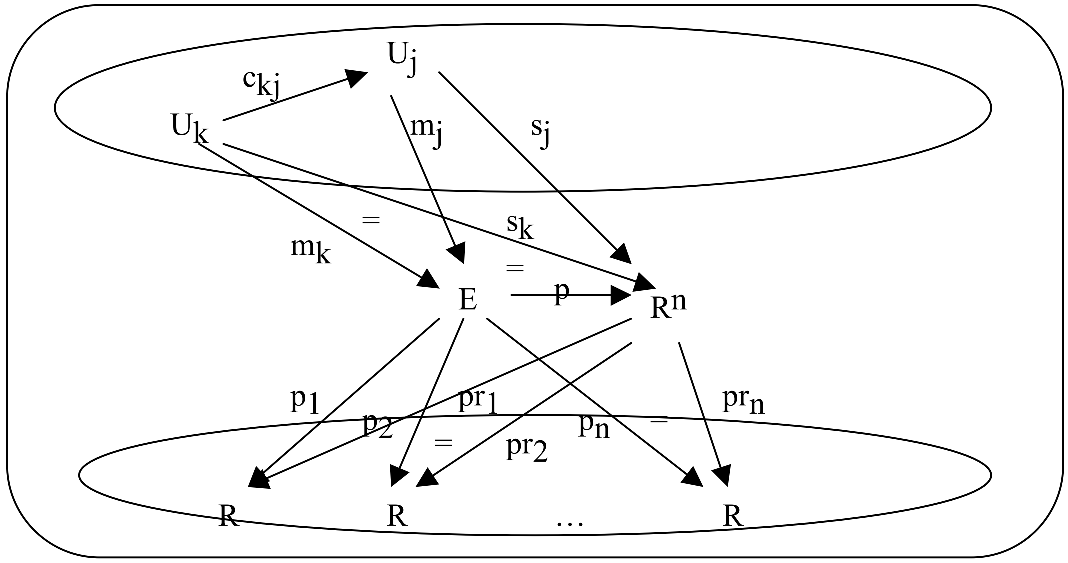

figure 9

Those descriptions are possible only now in the XXth century, when the notion of an arbitrary map and of the category of sets have become familiar. As expressed here, the accent is put on the ''universal property'' of the ''projective limit'', and this did not make sense before the 1950's. This led in the 1960's to the notion of a sketch defined by Ehresmann between 1960 and 1970 (Ehresmann, 1968, 1983), in the framework of the theory of categories.

In fact a relational system of coordinates (paragraph 3) could also be presented as a sketch, with underlying category a category of relations. So, as relational system or as morphisms toward a projective limit, and as realization of a sketch, general curvilinear coordinates on a space are now understable as a diagrammatical structure.

5- From figures and grid toward categories and sketches

In the first part we showed the signs by which the description of the classical sense of coordinates as grids merged progressively in the mathematical field.

In the second part we argued that, generally speaking and in the mathematical case, significations, interpretations, and signs could be understood as arrows and systems of arrows. Those arrows participate in an abridging process with maps.

In the third and fourth parts, we considered the modern mathematical setting of categories, as a very functionally oriented system of abbreviations by arrows. We show how in this setting « classical » introductions of coordinates as grids are metamorphosed into functional systems of maps and matrices, namely as relational systems of coordinates and as sketches.

So nowadays a coordinate system is an abstract diagrammatic articulation of an object E in the framework of other objets in the same category, an incorporation of E as a source of a sign which is an arrow p.

If we compare the part played by this coordination p in figure 4 and equations (8), to the part played by p in figure (9) and equation (15), we see that in the first case, p is *external* and full (under this name is subsumed a content of figures and equations), and in the second case, p is *internal* and scooped out (it is an element acting in an environment of figures and equations). So, thinking in terms of arrows allowed the metamorphosis of the picture fig. 4 into the sketch fig. 9.

The radical change is that all the signs describing initially the substantial components inside the grid are now put outside in order to sketch the functional property of coordination.

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