Klein's Group as a Borromean Object*

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Summary : Initially inspired by the case of the standard borromean link, we introduce the notion of a borromean object in a category. We provide examples in groups, boolean algebras, semi-rings, rings, fields. But the notion is introduced mainly for the case of the famous Klein's group $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$, that we describe as a borromean object in groups.

Résumé : Inspiré d'abord par le cas classique de l'entrelac borroméen, nous introduisons la notion d'objet borroméen dans une catégorie. Nous donnons des exemples dans les groupes, les algèbres de Boole, les semi-anneaux, les anneaux, les corps. Mais la notion est introduite surtout pour comprendre la structure du fameux groupe de Klein $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$, comme en effet un objet borroméen dans la catégorie des groupes.

Key words : borromean object, borromean rings, Klein's group, Klein's quartic.

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This paper is a continuation of the paper [6] (Amiens, 2005). It is the expanded version of some of the results announced in the conference [7] (Calais, 2008).

1 Borromean diagram, circular borromean algebra

Roughly speaking a *borromean diagram* for an object B is a presentation of this object B as a glueing of three components R, S, I such that if one of the three is eliminated, then the resulting situation is just a trivial composition of the other two objects. More precisely:

Definition 1 [borromean diagram] Let $\mathcal{C}$ be a category with null morphisms, cokernels, and a bi-functor $T : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$. A *borromean diagram* for an object B in $\mathcal{C}$ relatively to T consists of three objects R, S, I in $\mathcal{C}$ and an epimorphic family of monomorphisms in $\mathcal{C}$, $r : R \to B, s : S \to B, i : I \to B$ such that $B/r \simeq T(S, I)$, $B/s \simeq T(I, R)$, $B/i \simeq T(R, S)$. Given such a diagram for B,

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we say that B is a *borromean object*.
Two usual special cases are with $T(X,Y) = X + Y$ or $T(X,Y) = 1$:

Definition 1s [standard borromean diagram] Let $\mathcal{C}$ be a category with null morphisms, cokernels and finite sums. A *standard borromean diagram* for an object B in $\mathcal{C}$ consists of three objects R, S, I in $\mathcal{C}$ and an epimorphic family of monomorphisms in $\mathcal{C}$, $r : R \to B, s : S \to B, i : I \to B$ such that $B/r \simeq S + I$, $B/s \simeq I + R$, $B/i \simeq R + S$.

Definition 1r [reduced borromean diagram] Let $\mathcal{C}$ be a category with null morphisms, cokernels and terminal object 1. A *reduced borromean diagram* for an object B in $\mathcal{C}$ consists of three objects R, S, I in $\mathcal{C}$ and an epimorphic family of monomorphisms in $\mathcal{C}$, $r : R \to B, s : S \to B, i : I \to B$ such that $B/r \simeq 1$, $B/s \simeq 1$, $B/i \simeq 1$.

Remark 1 : *We do not assume in these definitions the extra condition that* each of the families (r,s), (s,i) and (i,s) is not epimorphic*. We let the study of this more strict structure for another occasion.*

Furthermore in the idea of a borromean diagram we could assume that the three components are isomorphic and trivial and that they play similar parts, and then we speak of a *circular borromean diagram.* Precisely:

Definition 2 [circular borromean algebra] In a variety of Ω-algebras with unit (e.g. in groups (with 1 as unit), rings (with 0 as unit), lattices (with 0 as unit), etc.)[1, p. 162], an object B is a *circular borromean algebra* (*cba*) if and only if the two following conditions hold:
1) $B \simeq F(r,s,i)/R$ with $F(r,s,i)$ the free algebra on three generators r, s and i, with R a congruence invariant by the cyclic permutation $r \mapsto s \mapsto i \mapsto r$.
2) $B/(r=1) \simeq E_0(s,i)$, $B/(s=1) \simeq E_0(i,r)$, $B/(i=1) \simeq E_0(r,s)$, where $E_0(u,v)$ is a given algebra generated by two generators u and v (i.e. a given quotient of the free algebra on two generators $F(u,v)$, which is thought as "the" easy or trivial glueing of u and v).
Mainly we consider two cases for $E_0(u,v)$:
(1) $E_0(u,v) = F(u,v)$ (the free algebra of rank two): we get a case of Definition 1s.
(2) $E_0(u,v) = F(u,v)/(u = v = 1) = T$ (the terminal algebra): we get a case of Definition 1r.

Remark 2 [analogue of Remark 1] : *We do not assume in these definitions the extra condition that in B the structure generated by* each of the pairs (r,s), (s,i) or (i,s) is not B*. So this condition is not satisfied for G_{168}.*

2 Borromean links, and some other examples

Proposition 1 [The group of the borromean link as a borromean group] *If we look at the group of the ordinary standard borromean link, its computation by the method of Dehn provides its circular generation by* r, s, i *with the relations*

$$rir^{-1}sr = srs^{-1}is = isi^{-1}ri.$$

In this way this group is a circular borromean group (as in definition 2, case (1)). And this fact expresses exactly the borromean property of the link itself. Clearly if we put $r = 1$ we get $is = is = is$, that is to say no conditions, and the resulting group is the free group of rank two. Furthermore it is known that if the group of a link is a free group of rank 2, then the link is a trivial link with two components (see [9, p. 74]).

Proposition 2 [The borromean group $\mathcal{S}(3)$]. *The group* $\mathcal{S}(3)$ *is a circular borromean group (as in definition 2, case (2)), generated by* u, v, w, *with the relations*

$$u^2 = v^2 = w^2 = 1, \quad uv = vw = wu, \quad vu = uw = wv.$$

We consider that

$$u = (12), \quad v = (23) \quad w = (31),$$

$$uv = vw = wu := c^+ = (123), \quad vu = uw = wv := c^- = (132).$$

And if we add $u = 1$ then we get $w = vw$, $v = 1$, and $w = uv$, $w = 1$.

Proposition 3 [The borromean group $\mathbb{Z}/7\mathbb{Z}$]. *The additive group* $\mathbb{Z}/7\mathbb{Z}$ *is a circular borromean group (def. 2 case (2)) generated (in multiplicative notations) by* u, v, w, *with the relations*

$$u^2 = v, v^2 = w, w^2 = u, \quad uv = vu = w^6, vw = wv = u^6, \quad wu = uw = v^6.$$

We construct $\mathbb{Z}/7\mathbb{Z}$ in additive notations with $u = 1, v = 2, w = 4$. And if $u = 1$ we get $v = u^2 = 1$, $w = v^2 = 1$.

Proposition 4 [The non-borromean group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$]. *The additive group* $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ *admits a circular generation given by* u, v, w, *with, the relations*

$$u^2 = v^2 = w^2 = 1, \quad uv = vu = w, vw = wv = u, \quad wu = uw = v.$$

But this group is not *a circular borromean group (def. 2 case (2)).*
We construct $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in additive notations with $u = (0, 1), v = (1, 0)$,

and $w = (1, 1)$. It is, up to permutations, the only possible circular generation, and this one is not borromean: if we add $u = 1$, then we only get $v = w$ and $v^2 = 1$, but not necessarily $v = w = 1$.

In the next section we will see a more complex example of a borromean group, the case of the Klein's group. But before that, let us see some examples in other categories, namely in the categories of boolean algebras, of semi-rings, of rings, of fields. For example by a *borromean ring* we mean a borromean object in the category of rings! So for the classical topological object called "borromean link" but also called "borromean rings", we prefer the first name, the second one being now confusing.

Proposition 5 [A borromean boolean algebra] *Let E be a set, let A, B, C be a partition in three parts of E. Then the boolean algebra $\mathcal{P}(E)$ is equipped with a borromean diagram in the sense of Definition 1s, taking*

$$R = \mathcal{P}(A), \quad S = \mathcal{P}(B), \quad I = \mathcal{P}(C)$$

Conversely, a borromean diagram on $\mathcal{P}(E)$ determines a partition in three parts on E. So a borromean object is a natural extension of the Sesmat-Blanché hexagram (extending itself the Aristotle square).

Proposition 6 [A borromean sub semi-ring of $\mathrm{Mat}_3(\{F, T\})$] *We consider $\{F, T\}$ as a semi-ring, with the conjunction as multiplication and the disjonction $\vee$ as addition, and $\mathrm{Mat}_3(\{F, T\})$ is the associated semi-ring of square boolean matrices of order 3. In this semi-ring we consider (with $F = 0$ and $T = 1$)*

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In fact R, S, I generate by product the subgroup $\mathrm{Inv}\big(Mat_3(\{F, T\})\big)$ of invertible elements of $Mat_3(\{F, T\})$; the sub semi-ring $< R, S, I >$ of the semi-ring $Mat_3(\{F, T\})$ generated by R, S, I consists of 49 elements, and it is freely generated by R, S, I with the relations:

$$RR = SS = II, \quad RS = SI = IR, \quad SR = IS = RI,$$

$$R \vee S \vee I = R(R \vee S \vee I) = S(R \vee S \vee I) = I(R \vee S \vee I).$$

It is a circular borromean semi-ring.

Proposition 7 [The borromean ring $\mathrm{Mat}_2(\mathbb{F}_2)$] *The ring $\mathrm{Mat}_2(\mathbb{F}_2)$ is a circular borromean object in the category of rings (as in definition 2, case (2)), generated by r, s, i with the relations*

$$r + r = s + s = i + i = 0, \quad r^2 = s^2 = i^2 = 1,$$

$$rs = si = ir, \quad sr = is = ri, \quad r + s + i = 0.$$

Of course if we put $r = 0$, then $0 = 1 = r = s$, and, as expected the structure becomes trivial. In order to generate $\mathrm{Mat}_2(\mathbb{F}_2)$ we takes

$$r = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right], \ s = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right], \ i = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right].$$

These matrices satisfy the given relations. With $c^+ = rs = si = ir$ and $c^- = sr = is = ri$, the sixteen elements of $\mathrm{Mat}_2(\mathbb{F}_2)$ are :

$$0, 1, r, s, i, c^+, c^-, 1+r, 1+s, 1+i, r+c^+, s+c^+, i+c^+, r+c^-, s+c^-, i+c^-.$$

We verify that, because of the generating relations, each product on the left or on the right of these elements with r, s, i, c^+, c^- is again in the list, and the same for additions.

Proposition 8 [The borromean field $\mathbb{F}_8$]. *The field $\mathbb{F}_8$ with 8 elements could be presented as $\mathbb{F}_8 = \{0, 1, a, b, c, a^{-1}, b^{-1}, c^{-1}\}$, with the relations*

$$a + a = b + b = c + c = 0, \quad ab = ba, \quad bc = cb, \quad ca = ac,$$

$$abc = 1, \quad ab + bc + ca = 0, \quad a + b + c = 1,$$

$$a^{-1} = c + 1 = bc, \quad b^{-1} = a + 1 = ca, \quad c^{-1} = b + 1 = ab,$$

$$a^2 = b, \quad b^2 = c, \quad c^2 = a,$$

$$a + a^{-1} = b, \quad b + b^{-1} = c, \quad c + c^{-1} = a.$$

and so it is a circular borromean field.

This presentation of $\mathbb{F}_8$ is used in [6]. In fact a, b and c are the roots of $X^3 + X + 1 = 0$, and a^{-1}, b^{-1}, c^{-1} are the roots of $X^3 + X^2 + 1$.

3 The circular borromean group $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$

In a previous paper [6] we proved that for every n and every k, every function $\left(\mathbb{F}_{2^n}\right)^k \to \mathbb{F}_{2^n}$ is a composition of constants, $\wedge, \neg$ and $(-)^2$, where $(\wedge, \neg)$ is a boolean structure on $\mathbb{F}_{2^n}$ associated to a normal basis, and $(-)^2$ the Frobenius map of the field. In fact the Frobenius is expressible as a composite of the given constants, $\wedge, \neg$, with the $\wedge_i, \neg_i$, $i = 1, 2, 3$ for three precise other boolean structures on $\mathbb{F}_{2^n}$, isomorphic to the first one, *but* different of it. In the case of $\mathbb{F}_8$ (i.e. $n = 3$) an explicit symmetric solution exists, which is associated to three special bases of $\mathbb{F}_8$ over $\mathbb{F}_2$ given by the three matrices

$$r = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},\ s = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},\ i = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

which are, with respect to the unique normal basis, the matrices of the three other strictly auto-dual bases. These facts are proved (fully although in a rather compact way) in [6]. So are introduced computations in *Moving Logic* (cf. also [5]). It was for such a logical calculus that the three elements r, s and i were introduced. But now we would like to leave the area of logic, and to concentrate our attention on these r, s and i and there significance with respect to the symmetry of the structure of the group $\mathrm{GL}_3(\mathbb{F}_2)$ (for the counterpart in the symmetry of Moving Logic, see [6, théorème 5]).

This group $\mathrm{GL}_3(\mathbb{F}_2)$ is well known as being isomorphic to G_{168} (the only simple group with 168 elements), which appeared in the work of Felix Klein [10] in 1879, as $\mathrm{PSL}_2(\mathbb{F}_7)$, and also as the group of homographies of $\mathrm{P}_2(\mathbb{C})$ which let invariant the Klein's quartic

$$\mathrm{X}(7) = \{[x : y : z] \in \mathrm{P}_2(\mathbb{C})\ ;\ x^3y + y^3z + z^3x = 0\}.$$

A lot of informations on $\mathrm{X}(7)$ and its group $G_{168} \simeq \mathrm{PSL}_2(\mathbb{F}_7) \simeq \mathrm{GL}_3(\mathbb{F}_2)$ are available in the book [11]. In fact $\mathrm{X}(7)$ is a smooth algebraic curve, and so is riemannian, and its genus is 3. Its group of homographic symmetries G_{168} is the maximal group in genus 3 (see [8]). On $\mathrm{X}(7)$ we can inscribe a borromean link without double points, with a nice ternary symmetry, and so the borromean link is of genus 3. It was the reason for which we would like to understand $\mathrm{X}(7)$ or its group G_{168} as itself a borromean object in a convenient category. In fact we will see here that the system of the r, s , i is an algebraic analogue *in* G_{168} of the borromean link in $\mathrm{X}(7)$.

Proposition 9 [transposition by conjugaison]. *1 — We have* $r^{-1} = r^6$, $s^{-1} = s^6$, $i^{-1} = i^6$, *and*

$$r^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ s^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ i^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

2 — The transposed r^t, s^t, i^t *of the matrices* r, s, i *are given by :*

$$r^t = rir^{-1},\quad s^t = srs^{-1},\quad i^t = isi^{-1}.$$

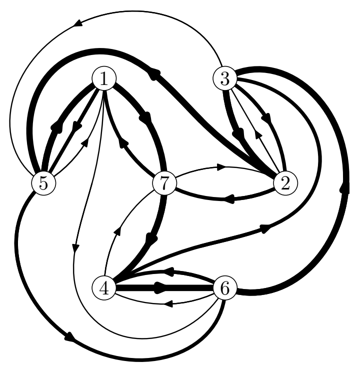
Proposition 10 [Circular symmetry among r, s, i] *There is a representation of* $\mathrm{GL}_3(\mathbb{F}_2)$ *in* $\mathcal{S}(7)$ *such that*

$$r = [1746325], \quad s = [1647235], \quad i = [1564327],$$

and such that, with $j = (142)(356)$ *we get:*

$$jrj^{-1} = s, \quad jsj^{-1} = i, \quad jij^{-1} = r,$$

and this situation could be observed on the following figure:



This in fact is already proved in [6], looking to the left action of r, s, i on the columns $1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, ..., $7 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The figure shows concretely this circular symmetry, algebraically realized by the conjugaison by j.
NB : j is not in the image of $\mathrm{GL}_3(\mathbb{F}_2)$ in $\mathcal{S}(7)$.

Proposition 11 [The circular borromean group $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$].
1 — The group $\mathrm{GL}_3(\mathbb{F}_2)$ *is generated by* r*,* s *and* i *above, with,* among others, *the relations*

$$(srs^{-1}rir^{-1})^2 = 1, \quad (is^3i^{-1})^7 = 1,$$
$$((is^3i^{-1})(srs^{-1}rir^{-1}))^3 = 1, \quad ((is^3i^{-1})^4(srs^{-1}rir^{-1}))^4 = 1.$$

2 — $r^7 = s^7 = i^7 = 1$.
3 — Furthermore if $w(r, s, i) = 1$ *is satisfied, with* $w(r, s, i)$ *any word in* r*,* s*,* i*, then also* $w(s, i, r) = 1$*,* $w(i, r, s) = 1$.
4 — If in the group $\mathrm{GL}_3(\mathbb{F}_2)$ *one (e.g.* r*) of the three generators* r*,* s *or* i *is cancelled (by adjonction of* $r = 1$*) then the quotient group reduces to the trivial one.*
So the group $\mathrm{GL}_3(\mathbb{F}_2)$ *is a* circular borromean algebra *in the sense of the definition 2, case (2). We speak also here of a* borromean spanning *of* $\mathrm{GL}_3(\mathbb{F}_2)$.

In [6] this result is only announced (Theorem 3 in [6]), without proof, and in fact with a mistake (the srs^{-1} were unfortunately replaced by s).

For a proof now we know [2, p.303] that the group $\mathrm{GL}_3(\mathbb{F}_2)$ is freely generated modulo Dyck's relations ([3, p.41]):

$$T^2 = I,\ S_1^7 = I,\ (S_1T)^3 = I,\ (S_1^4T)^4 = I,$$

by
$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \text{ and } S_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

So, in order to conclude for the first point, we need only to observe that

$$rs = T^t, \quad i^3 = S_1^t,$$

and then, with Proposition 9 we get

$$T = (rs)^t = s^t r^t = srs^{-1}rir^{-1}, \quad S_1 = (i^t)^3 = is^3i^{-1},$$

that is to say:

$$T = srs^6rir^6, \quad S_1 = is^3i^6.$$

For the next part of the proposition, it is a consequence of Proposition 10. The last affirmation is just a consequence of the fact that $\mathrm{GL}_3(\mathbb{F}_2)$ is simple; it is also an easy consequence of the relations given in (*1*—) *and* (*2* —).
Let us remark that the relations (*2* –) (r, s and i are of order 7) cannot be deduced from the relations (*1* —) (which come from the Dyck's relations). Without these relations (*2* —), the group generated under relations (*1* —) is not G_{168}, because its quotient by $r = 1$ is not trivial and so the group is not simple. But if the relations (*2* —) could be deduced from one, then the quotient by $r = 1$ would be trivial.

Of course an analogue of this proposition is possible starting from any system of generators of G_{168}. For example:
Proposition 12 [for a variant of proposition 11]. *Starting with the generation of* $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$ *given* (cf. [12]) *by the two generators*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

we could conclude that r, s, i *generate* G_{168}*, as we have:*

$$A = srisr^2sris, \quad B = sris.$$

Proposition 13 [The generation of G_{168} is not strict] *The circular borromean generation of* G_{168} *by the* r, s, i *is not strict, in the sense that it is false that* G_{168} *is not generated by* r, s*, or by* s, i *or by* i, r*. More precisely we have:*

$$r = s^4i^6s, \quad s = i^4r^6i, \quad i = r^4s^6r.$$

But also we can obtain our result (that G_{168} is a circular borromean group) directly, without Dyck's or Müller-Ritzenthaler's results, in the following way.

Proposition 14 *In* $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$ *we have an isomorphic image*

$$\mathcal{S} = \{Id_3, U, V, W, C^+, C^-\}$$

of $\mathcal{S}(3)$ *(as presented in* Proposition 2*) with*

$$U = ri, \;\; V = sr, \;\; W = is,$$

$$C^+ = UV = VW = WU, \;\; C^- = VU = UW = WV.$$

To u, v and w in $\mathcal{S}(3)$ *are associated U, V, W in* $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$ *.*

Proposition 15 *In* $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$ *we have an isomorphic image*

$$\mathcal{H} = \{Id_3, R, S, I, R^{-1}, S^{-1}, I^{-1}\}$$

of $\mathbb{Z}/7\mathbb{Z}$ *(as presented in* Proposition 3*) with*

$$R = ir^2, \;\; , S = rs^2 \;\; , I = si^2, \;\; R^{-1} = i^3sr, S^{-1} = r^3is, \;\; I^{-1} = s^3ri.$$

To u, v and w in $\mathbb{Z}/7\mathbb{Z}$ *are associated R, S, I in* $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$*.*
So $\mathcal{H}$ *is also an isomorphic image of the group of invertible elements of the borromean field* $\mathbb{F}_8$*, with R, S, I the images of the a, b, c (*Proposition 3*). In fact* $\mathcal{H}$ *is a Fano plane, i.e. a model of the projective plane* $\mathrm{P}_2(\mathbb{F}_2)$*, in which the seven lines are:*

$$R^\perp = \{S^{-1}, S, I\}, S^\perp = \{R, I^{-1}, I\}, I^\perp = \{R, S, R^{-1}\},$$

$$(S^{-1})^\perp = \{R, S^{-1}, Id_3\}, (I^{-1})^\perp = \{S, I^{-1}, Id_3\}, (R^{-1})^\perp = \{I, R^{-1}, Id_3\},$$
$$(Id_3)^\perp = \{S^{-1}, I^{-1}, R^{-1}\}.$$

It is known that the incidence graph of $\mathrm{P}_2(\mathbb{F}_2)$, as it could be described from the end of the previous Proposition 15, is the Heawood's graph Hwd, and furthermore Aut(Hwd) $\simeq \mathrm{PGL}(2,7)$ is of order 336 and contains G_{168} as a subgroup of index 2. From that we easily obtain:

Proposition 16 [G_{168} as the group of automorphisms of a graph] *The Heawood graph* Hwd *consists of 14 summits on a circle in the order*

$$0, 2', 1, 3', 2, 4', 3, 5', 4, 6', 5, 0', 6, 1', 0,$$

linked in this order, with the additional links $04'$, $15'$, $26'$, $30'$, $41'$, $52'$, $63'$. *We augment it as* Hwd^{+} *with 7 new summits* $0'', 1'', 2'', 3'', 4'', 5'', 6''$ *and the links* $00'', 11'', 22'', 33'', 44'', 55'', 66''$. *Then* $G_{168} = \mathrm{Aut}(\mathrm{Hwd}^{+})$.

These facts suggested that it could be possible to generate G_{168} as a kind of extension of its subgroup $\mathcal{H}$, as we will do now.

NB: [Notation] *Do not confuse in* $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$ *the elements* R, S *and* I *with the elements* r, s, i. *We introduce the notation:*

$$\mathcal{B} = \{Id_3, r, s, i\}.$$

Proposition 17 *The group* $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$ *is a circular borromean group, generated by* r, s, i; *and more precisely, every element* m *of* $\mathrm{GL}_3(\mathbb{F}_2)$ *could be written in a unique way as a composition*

$$m = hbk, \text{with} \quad h \in \mathcal{H}, b \in \mathcal{B}, k \in \mathcal{S},$$

and so every m *is a word in* r, s, i *of length less or equal to 10.*

In order to prove that $\mathrm{GL}_3(\mathbb{F}_2) = \mathcal{HBS}$, we compute the 28 compositions hb, with $h \in \mathcal{H}$ and $b \in \mathcal{B}$, and we get the following values:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, r = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, s = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, i = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, Rr = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, Rs = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, Ri = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, Sr = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, Ss = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, Si = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

$$I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, Ir = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, Is = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, Ii = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$R^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, R^{-1}r = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, R^{-1}s = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, R^{-1}i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$S^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, S^{-1}r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, S^{-1}s = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, S^{-1}i = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

$$I^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, I^{-1}r = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I^{-1}s = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, I^{-1}i = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

By a simple inspection we see that these 28 matrices are really 28 distinct objects, and that each one is associated exactly to one of the 28 non-order

bases of the $\mathbb{F}_2$-field $\mathbb{F}_2^3$. So, to conclude and to get the 168 elements of the group $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$, we have just to produce the permutations of columns by a multiplication on the right with one of the 6 elements of $\mathcal{S}$.
The final point then is a consequence of the facts that elements of $\mathcal{H}$ are written with words of length at most 5, and elements of $\mathcal{S}$ are written with words of length at most 4.

And to conclude, let us indicate that the system r, s, i is not at all unique as a circular borromean generation of $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$. At least we have a kind of *mirror image* of it:

Proposition 18 [another circular borromean generation of $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$]
1 — *In* $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$ *we introduce (with the help of the description of transposition in* Proposition 9*) three new elements :*

$$A = r^t i^t, \quad B = s^t r^t, \quad C = i^t s^t,$$

with

$$r^t = rir^{-1}, \quad s^t = srs^{-1}, \quad i^t = isi^{-1}.$$

Then we have

$$ir = A^t, \quad rs = B^t, \quad si = C^t,$$

$$r = ACB, \quad s = BAC, \quad i = CBA.$$

2 — In the representation in $\mathcal{S}(7)$ *(*Proposition 10*)* A, B, C *are:*

$$A = (46)(57), \quad B = (23)(67), \quad C = (15)(37).$$

We have

$$A^2 = B^2 = C^2 = 1,$$

and so this generation is not isomorphic to the generation by the r, s, i.
3 — In the representation by matrices in $\mathrm{GL}_3(\mathbb{F}_2)$ *the elements* A, B, C *are the three basic transvections:*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4 — A, B, C *is a circular borromean presentation of* $G_{168} = \mathrm{GL}_3(\mathbb{F}_2)$.

Just we remark that the transvections A, B, C generate $\mathrm{SL}_3(\mathbb{F}_2) = \mathrm{GL}_3(\mathbb{F}_2)$ (see [4, p. 94]), and so we get a third proof that r, s, i generate $\mathrm{GL}_3(\mathbb{F}_2)$. Furthermore it is borromean, because if we add $A = 1$, then $ir = A^t = 1^t = 1$, and also $r = CB = i$, so $r^2 = 1$, and with $r^7 = 1$ we get $r = 1$.

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