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Logica Universalis

A Hexagonal Framework of the Field $\mathbb{F}_4$ and the Associated Borromean Logic

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Friendly dedicated to Jacques Riguet, on the occasion of his 90th birthday

Abstract. The hexagonal structure for 'the geometry of logical opposition', as coming from Aristoteles–Apuleius square and Sesmat–Blanché hexagon, is presented here in connection with, on the one hand, geometrical ideas on duality on triangles (construction of 'companion'), and on the other hand, constructions of tripartitions, emphasizing that these are exactly cases of borromean objects. Then a new case of a logical interest introduced here is the double magic tripartition determining the semiring $\mathcal{B}_3$ and this is a borromean object again, in the heart of the semiring $\mathrm{Mat}_3(\mathbb{B}_{\mathrm{Alg}})$. With this example we understand better in which sense the borromean object is a deepening of the hexagon, in a logical vein. Then, and this is our main objective here, the Post-Mal'cev full iterative algebra $\mathbb{P}_4 = \mathbb{P}(\mathbb{F}_4)$ of functions of all arities on $\mathbb{F}_4$, is proved to be a borromean object, generated by three copies of $\mathbb{P}_2$ in it. This fact is induced by a hexagonal structure of the field $\mathbb{F}_4$. This hexagonal structure is seen as precisely a geometrical addition to standard boolean logic, exhibiting $\mathbb{F}_4$ as a 'boolean manifold'. This structure allows to analyze also $\mathbb{P}_4$ as generated by adding to a boolean set of logical functions a very special modality, namely the Frobenius squaring map in $\mathbb{F}_4$. It is related to the splitting of paradoxes, to modified logic, to specular logic. It is a setting for a theory of paradoxical sentences, seen as computations of movements on the bi-hexagonal link among the 12 classical logics on a set of 4 values.

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1. Logics Coming from Borromean Objects

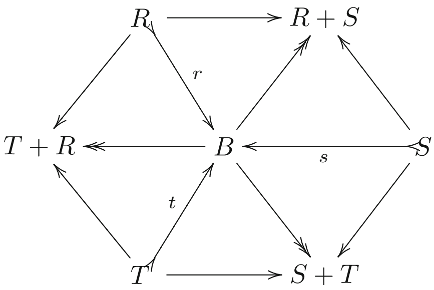
The idea of a hexagonal setting in logic of opposition could be deepened, in a 'functional' direction (i.e. in order to show and analyze algebras of functions) by the notion of a borromean object, recalled here.

Definition. A standard (or resp. a reduced) *borromean object* in a category $\mathcal{C}$ with null morphisms (and a terminal and initial object), cokernels and finite sums (finite coproducts), is (definition 1s-1r, [11, p. 145]) an object B equipped with three objects R, S, T in $\mathcal{C}$ and an epimorphic family of monomorphisms in $\mathcal{C}$,

$$r : R \to B, \quad s : S \to B, \quad t : T \to B$$

such that $B/r \simeq S + T, B/s \simeq T + R, B/t \simeq R + S$ (or resp. such that $B/r \simeq 1, B/s \simeq 1, B/t \simeq 1$).

So the standard case is pictured in $\mathcal{C}$ as β_B:



In the category of finite boolean algebras this notion is equivalent to a tripartition of a set (Proposition 3.4). We have also the case of pointed 3-partitions (Proposition 3.5). In the category of semi-rings, a logically meaningful example is the semi-ring $\mathcal{B}_3$ (Proposition 4.1). In the category of groups we get the examples of the fundamental group of the complement of a *borromean link*, of the groups $\mathcal{S}(3)$, $\mathbb{Z}/7\mathbb{Z}$, and—more sophisticated—the Klein's group G_{168} of the Klein's quartic (cf. [11]).

Of course the diagram of a borromean object has basically a hexagonal framework, but its logico-geometrical scheme β is deeper than the simple picture of a hexagon. Especially the 'opposition' in a borromean diagram does not construct the opposite as a complement, but as a quotient or a dual. Furthermore the central object B is not determined by its components R, S, T, and in fact has to be thought as a new datum, the datum of an original link between the components.

It happens that if the objects of $\mathcal{C}$ have a 'logical meaning' (e.g. if they are boolean algebras), then a given borromean object B in $\mathcal{C}$ generates a logico-geometrical borromean system of such 'logical meanings'—denoted by $\beta[B]$—and such a mixture is a kind of cross product of the pure abstract logico-geometrical scheme β with the inner logical content of each object in the diagram of B. We will show in detail such a structure in the case of the tripartition of a set, and in the case of the semi-ring $\mathcal{B}_3 \subset \mathrm{Mat}_3(\mathbb{B}_{\mathrm{Alg}})$, corresponding to a special double tripartition of the set $\{1,2,3,4,5,6,7,8,9\}$.

An application of this approach is given (with a short but complete proof) in [7] to a borromean analysis of any finite field $\mathbb{F}_{2^n}$, and (without proof) in the cases $n = 2$ and $n = 3$.

Now in the case of $\mathbb{F}_4$ we will proceed to a very effective construction. We let the details of the case $\mathbb{F}_8$, and the related structure of $\mathrm{Mat}_3(\mathbb{B}_{\mathrm{Alg}})$, for another publication.

Our main result could be formulated as follows: the hexagonal and more precisely the borromean structure of $\mathbb{F}_4$ provides three well determined structures of classical boolean logic on $\mathbb{F}_4$ (those of which the false is 0). The logical functions corresponding to these three structures generate the system of all functions via composition. Furthermore, composition of one special system of logical functions with the Frobenius'map $(-)^2$ also generates this system of all functions. This Frobenius' map appears as a 'cyclic non-logical modality' expressing a galoisian undiscernability (as a generator of the Galois'group of $\mathbb{F}_4$ over $\mathbb{F}_2$). Via this Frobenius'map we can explain how in $\mathbb{F}_4$ a calculus of paradoxes in $\mathbb{F}_2$ is possible.

We expand also the calculus of *logical speculations* or 'points of view' inside $\mathbb{F}_4$, and so the Frobenius undiscernability could be analyse as a difference or a sum of 'points of view'. Furthermore these points of view are twelve and organized according to a bi-hexagonal picture.

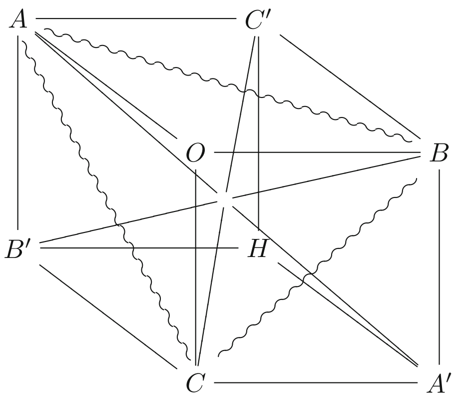
We show explicitly how to obtain the Webb–Sheffer's function $\Uparrow$ on $\mathbb{F}_4$, and also the multiplication of the field $\mathbb{F}_4$. So the arithmetic in $\mathbb{F}_4$ proceeds from a hexagonal or borromean glueing of a system of boolean structures (or also from a boolean structure equipped with a cyclic permutation among its atoms); such a datum is a case of a *boolean logical manifold.*

2. Triangle, Square and Hexagon in a Cube, and Borromean Link

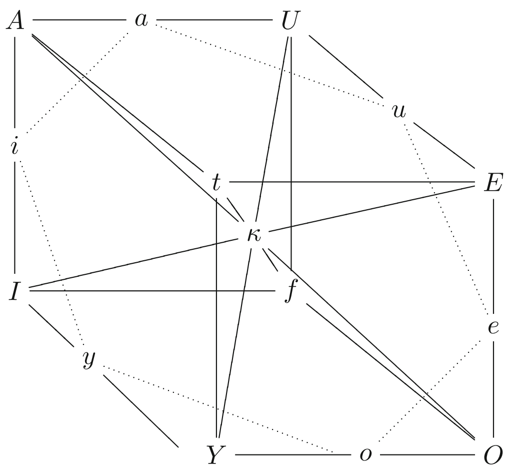
Here—and in the next section, starting from a pure geometrical setting, we analyze some ways along which we can move and change our diagrammatical presentations of data between *bi*, *tri*, *tetra*, *hexa* and *octo* systems of positions. In such a visual practice, with geometry as well as with logic, we get the basic skill with mathematical and metaphorical use of hexagonal data and borromean links. So we get geometric passages and a lot of geometric shapes in an equal right as possible frameworks for analysis of $\mathbb{F}_4$ as a logical manifold.

Proposition 2.1. *An arbitrary triangle ABC in the euclidean metric plane is naturally equipped with a* companion *$A'B'C'$, both parts of a hexagon $AC'BA'CB$ with parallel opposite sides, A', B' and C' being the symmetrics of the center O of the circle $\Gamma(A,B,C)$ circumscribed to ABC with respect to BC, CA and AB. Then the orthocenter H of ABC becomes the center of $\Gamma(A',B',C')$, and all that makes up the shadow of a parallelepiped in which ABC is inscribed.*

Extending the picture given in [9, p. 161, Fig. 12a] we get:

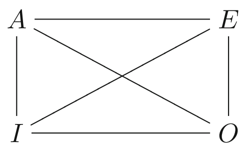


Proposition 2.2. *Any hexagon* $AUEOYI$ *with parallel opposite sides could be seen as the outline of the shadow of a cube with center* κ*, with a big diagonal* ft *drawn inside the shadow; and also the plane section* $aueoyi$ *of the same cube, orthogonally to the diagonal* ft *at the point* κ*, is a (regular) hexagon. If we start with a cube and choose a long diagonal, then we get a* transversal hexagon $aueoyi$*, which could be twisted to be assimilated to the outside hexagon* $AUEOYI$*. So, when you hold a cube between two fingers by two opposite corners, what you see is the hexagonal diagram.*



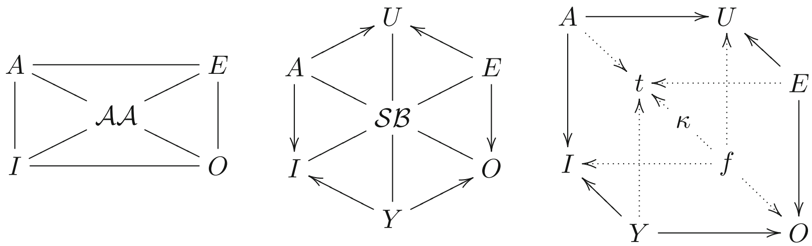
Proposition 2.3. *The logical square of Aristoteles–Apuleius* $\mathcal{AA}$ *is included in a logical hexagon of Sesmat–Blanché* $\mathcal{SB}$*, and this logical hexagon is inscribed in a* boolean logical cube $\mathcal{P}(\{a, b, c\})$ *with corners all the subsets of* $\{a, b, c\}$.

The famous Aristoteles–Apuleius logical square

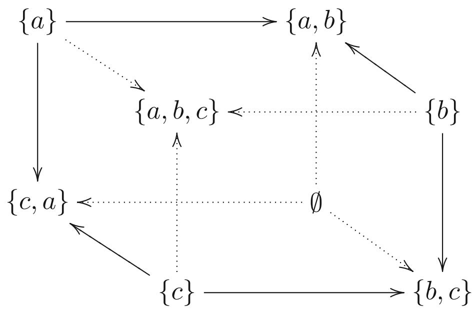


is based on the three aristotelian oppositions: contradiction, contrariety, subcontrariety. As the last step of a very long story of commentaries, recently these oppositions have been related by Béziau [1] to the three negations: classical, paracomplete, paraconsistent.

In the fifties it had been extended by Sesmat [17] and Blanché [2,3] to a logical hexagon, by adjunction to the old four positions A, E, I O, of two new positions Y and U, and so we get the Sesmat–Blanché hexagon, in which, for further use, we provide convenient orientations for edges; and so this hexagon could be seen as a part of a cube with ft as a long "inner" diagonal, with center κ.

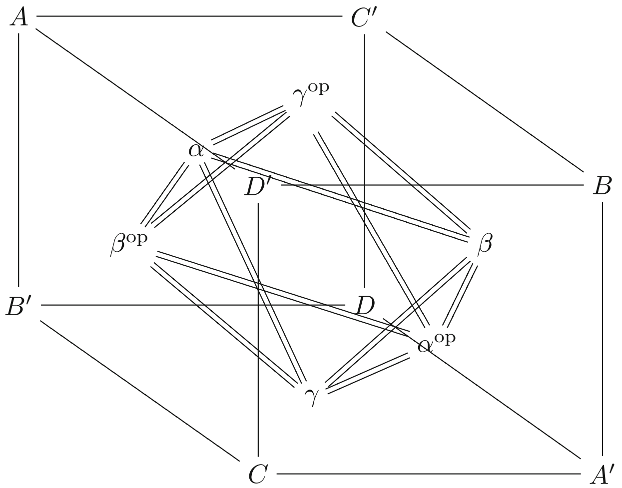


The cube κ is nothing else than the shape of a boolean logical cube $\mathcal{P}(\{a, b, c\})$:



Proposition 2.4. *The hexagon is not only a part or a quotient of a cube, it is also a graphical representation of the octahedron (which is dual of the cube).*

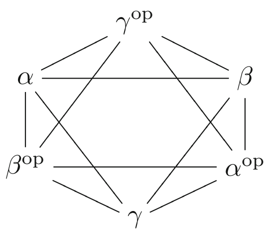
If we consider a cube as $\mathcal{P}(\{a,b,c\})$ and its dual octahedron, in which vertices are faces of the cube, linked by their contacts along edges, we get



with $A = \{a\}, B = \{b\}, C = \{c\}, D = \emptyset$, and with the opposite points $A' = \{b,c\}, B' = \{c,a\}, C' = \{a,b\}, D' = \{a,b,c\}$, where the faces of the cube $\mathcal{P}\{a,b,c\})$ are named as:

$$\alpha = \{\emptyset, \{a\}, \{a,b\}, \{c,a\}\} = \{D, A, C', B'\},$$
$$\beta = \{\emptyset, \{b\}, \{b,c\}, \{c,a\}\} = \{D, B, A', C'\},$$
$$\gamma = \{\emptyset, \{c\}, \{c,a\}, \{b,c\}\} = \{D, C, B', A'\},$$
$$\alpha^{\rm op} = \{\{a,b,c\}, \{b,c\}, \{c\}, \{b\}\} = \{D', A', C, B\},$$
$$\beta^{\rm op} = \{\{a,b,c\}, \{c,a\}, \{a\}, \{c\}\} = \{D', B', A, C\},$$
$$\gamma^{\rm op} = \{\{a,b,c\}, \{a,b\}, \{b\}, \{a\}\} = \{D', C', B, A\}.$$

From the previous picture we extract the incidence picture of the octahedron:



Proposition 2.5. *The octahedral hexagon in Proposition 2.4 could also been considered as producted by a tetrahedron, with the identifications of the six vertices as the six edges of the tetrahedron $ABCD$: $\alpha = AD$, $\beta = BD$, $\gamma = CD$, $\alpha^{\rm op} = BC$, $\beta^{\rm op} = AC$, $\gamma^{\rm op} = AB$. This could also be realized with $A'B'C'D'$, and it is related to the fact that a cube is a bi-tetrahedron* (*see Proposition 2.7*).

Proposition 2.6. *Starting from a tetrahedron $ABCD$ we could obtain the hexagonal situation as in Proposition 2.5, and through this identification we get transfer from 4 to 3 and from 4 to 6.*

1. *We have a surjective homomorphism* $\delta : \mathcal{S}(4) \to \mathcal{S}(3)$*: looking at* $\mathcal{S}(4)$ *as the group of permutations on the vertices of a tetrahedron, it operates also on the three diagonals joining the middle points of opposite sides.*
2. *We get in injective homomorphism* $\iota : \mathcal{S}(4) \to \mathcal{S}(6)$*, by the action of* $\mathcal{S}(4)$ *on the* 6 *edges of* $ABCD$*.*

So from a geometrical point of view, there are trivial and less trivial processes to travel from 3 to 6, from 6 to 8, from 8 to 6 to 4 and to 3, from 4 to 3 and 6, etc. Others constructions are possible. For example it is known that the datum of six points A, B, C, D, E and F in the complex projective plane is projectively equivalent to the data in the metric plane of a triangle C', D', E' and a point F': we just transform A and B in the cyclic points I and J, and then the metric is fixed. So we have here a process to transform 6 data in a given context to 4 data in another appropriated context.

But at this moment what we have to emphasize is that all these variations come geometrically from the introduction of the picture of 3, and then we reach pictures with $6, 4, 8$ (and also 12 and 24) elements, these modifications depending on variations of contexts as well as on pure geometrical tricks.

Then the question is the interest of such pictures in logic. The first idea is to use them as objects which are to be decorated by formulas in a calculus (showing relationship among formulas by a geometrical disposition, by properties of symmetry and duality), or even by concepts in an ideological construction, or in a modeling of a field of knowledge. What we get in such a presentation is that some dualities (for instance the negation in logic) could be presented twice, a first time as a function in the theory, and a second time as an operation on the theory.

Proposition 2.7. *Aristoteles–Apuleius square or Sesmat–Blanché hexagon are useful in logic, when they are* decorated *with formulas, relations or even objects* (*structures*) *of a logical interest, showing a hexagonal opposition and a dihedral symmetry among these data. The opposition so exhibited is not necessarily homogeneous with the decorations, not necessarily inherent to any of the decorating objects; it is a geometrical extra-modality. Among a large choice of interesting extensions on* $\mathcal{AA}$ *and* $\mathcal{SB}$*, the idea of bi-simplex and the general logic of* n-opposition *is especially important.*

These extensions and weak versions—obtained at first from the examination of the case $n = 3$ (and its double $2 \times 3 = 6$)—are introduced in papers like [14] and [15], by Alessio Moretti and Régis Pellissier. They will be necessary in future.

But our aim here is rather to deepen the understanding of the hexagon alone, of the visual thinking of 3-dim objects and logics, through the idea of a borromean object or a borromean diagram.

To complete the visual review of apparitions of a hexagon in the 3-dim game, we have to mention the *borromean link*. This link is not to be confused with the general idea of a *borromean object* (cf. Sect. 1), of which now it is just an example.

Proposition 2.8. *The picture of a hexagon has the same utility from a "logo-topic" point of view* (*see* [8, p. 5], [9, pp. 153–155]) *than the picture of a plane alternated projection of a borromean link of* 3 *circles* R, S, T (*the first in the Tait's series* [19]) *with equations*

$$R : x^2 + y^2 - 1 - \Big(- x + 0y\Big) = 0,$$

$$S : x^2 + y^2 - 1 - \Big(\frac{x}{2} + \frac{\sqrt{3}}{2}y\Big) = 0,$$

$$T : x^2 + y^2 - 1 - \Big(\frac{x}{2} - \frac{\sqrt{3}}{2}y\Big) = 0.$$

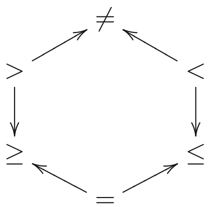
In A *and* A', R *goes over* S, *in* B *and* B', S *goes over* T, *in* C *and* C', T *goes over* R. *With* $r = \frac{\sqrt{17}+1}{4}$, $r' = \frac{\sqrt{17}-1}{4}$ *we have:*

$$A = \Big(- \frac{1}{2}, + \frac{\sqrt{3}}{2}\Big) r, \qquad A' = \Big(+ \frac{1}{2}, - \frac{\sqrt{3}}{2}\Big) r',$$
$$B = \big(+ 1, 0\big) r, \qquad B' = \big(- 1, 0\big) r',$$
$$C = \Big(- \frac{1}{2}, - \frac{\sqrt{3}}{2}\Big) r, \qquad C' = \Big(+ \frac{1}{2}, + \frac{\sqrt{3}}{2}\Big) r'.$$

3. Hexagon, Tripartition, Borromean Object

We emphasize the nature of a tripartition: it is a typical semantic for a Sesmat–Blanché hexagon (cf. Proposition 2.3), and it is also a case of a borromean object (definition in Sect. 1) in the category of boolean algebras.

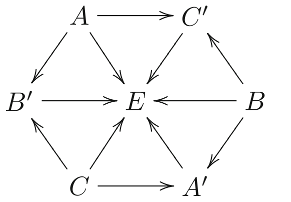
Proposition 3.1. *A typical case of the hexagon of Sesmat–Blanché is*:



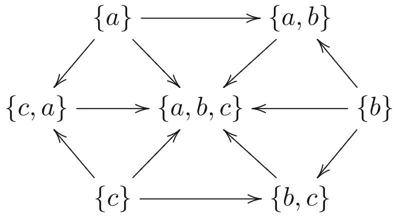
This case is exhibited and commented by Blanché; thinking of arrows as implications, we could understand the choice of orientation. If we are looking to order on real numbers, we know that we have three exclusive cases: $x < y$, $x = y$, or $x > y$; in fact the "logic" of the hexagon is concentrated on such a tripartition datum:

Proposition 3.2. *From a set theoretical point of view, the hexagon of Sesmat–Blanché describes the organisation of a generic* 3*-partition of an arbitrary set* E, *into* 3 *non-empty subsets* A, B, C *with* $A \cap B = B \cap C = C \cap A = \emptyset$ *and* $A \cup B \cup C = E$. *So, with* A', B' *and* C' *the complements of* A, B, C *we get* $A' = B \cup C = B + C$, $B' = C \cup A = C + A$, $C' = A \cup B = A + B$, *and in fact the properties of* A', B', C' *are dual of the ones of* A, B, C, *so we have*

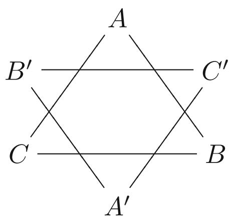
$A' \cup B' = B' \cup C' = C' \cup A' = E$ *and* $A' \cap B' \cap C' = \emptyset$, *and the following diagram of inclusions* $\mathcal{H}\{A, B, C\}$ *in the category* Set *of sets*:



Reducing the situation to the case $A = \{a\}$, $B = \{b\}$, $C = \{c\}$ *we get*:

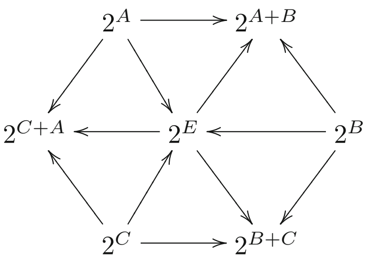


Proposition 3.3. *The star of David (which is to be included in the octahedron in Proposition 2.4)*



comes from the Sesmat–Blanché's hexagon when we stress on the bi-simplex aspect (i.e. on the choice of a diagonal in a cube), and on the opposition (contradiction) between the A, B, C *and* A', B', C'.

Proposition 3.4. *By direct and inverse extensions of these inclusions in the Proposition 3.2 to the powersets* 2^A, $2^B \dots$ *of* A, B, *etc., we get a diagram in the category* Bool *of boolean algebras:*



which exhibits 2^E *as a* borromean object *in* Bool, *in the sense of* [11] (*here see Sect.* 1). *This means that each diagonal is exact (e.g.* 2^{B+C} *is the quotient of*

2^E *by* 2^A*), and its last object is the sum of the two adjacent corner (e.g.* 2^{B+C} *is the sum of* 2^B *and* 2^C*).*

And conversely if these conditions in the category Bool *are satisfied, for boolean algebras of the type* 2^X*, then we get back a tripartition on a set, in the category* Set *of sets.*

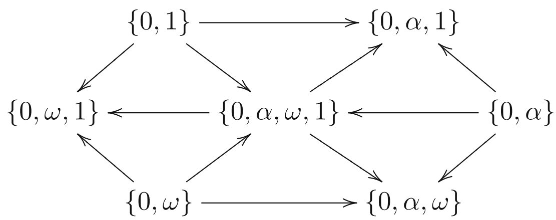
This proposition is given in [11], Prop. 5, p. 147. We could emphasize its meaning in the context of "hexagonal thinking". In the category of sets a logical tripartite organization (partition in 3) determines completely the central object, by glueing, and so a hexagonal picture in the Sesmat–Blanché style is enough; but in more structured categories the central object (for example a borromean link) is not determined by a glueing of its components, it is rather natural to think of such an object as being itself a *very original* glueing of its components. So we have not only one borromean link, but at least all the cases in the Taits series. So with these borromean cases in mind, we could add a central position to the Sesmat–Blanché hexagon.

Furthermore, the (opposition by) *negation* now is no longer a complementation, as in the formula $\{a, c\} = \{a, b, c\} - \{b\}$, but has to be understood as a representation of a quotient, as in the formula:

$$\mathcal{P}(\{a, c\}) = \mathcal{P}(\{a, b, c\})/\mathcal{P}(\{b\}).$$

To conclude this section, we can observed the following new simple example of a borromean object, useful in the analysis of the logic of $\mathbb{F}_4$ (at the beginning of Proposition 6.1).

Proposition 3.5. *The notion of borromean object works in the category of pointed sets—in such a case we will speak of* pointed borromean set*, or* pointed 3-partition *and so for a set with* 4 *elements* $\{0, \alpha, \omega, 1\}$*, pointed by* 0*, we get the borromean object picture*



4. Hexagonal or Borromean Aspects of $\mathbf{Mat_3(\mathbb{B}_{Alg})}$

In this section we construct a borromean object of logic interest, a semiring $\mathcal{B}_3$ of logical functions, related to a 'double' tripartition of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Let $\mathbb{B}_{\mathrm{Alg}}$ be the boolean semi-ring with 2 elements, that is to say the set $\{0, 1\}$ equipped with the two laws of intersection and union, denoted by $x \wedge y$ and $x \vee y$, with 0 and 1 the initial and final elements. The set of 3×3 matrices with coefficients in $\{0, 1\}$ becomes a semi-ring $\mathrm{Mat}_3(\mathbb{B}_{\mathrm{Alg}})$, with the laws $(mn)_{i,j} = \bigvee_k m_{i,k} \wedge n_{k,j}$, $(m \vee n)_{i,j} = m_{i,j} \vee n_{i,j}$.

Among its $2^9 = 512$ elements, we especially consider the three elements which generate by composition the group $(\mathrm{Mat}_3(\mathbb{B}_{\mathrm{Alg}}))_{\mathrm{inv}} \equiv S(3)$ of invertible elements of $\mathrm{Mat}_3(\mathbb{B}_{\mathrm{Alg}})$:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 4.1. *Let $\mathcal{B}_3 := \mathbb{B}_{\mathrm{Alg}}\{R, S, T\} \subset \mathrm{Mat}_3(\mathbb{B}_{\mathrm{Alg}})$ the sub-semi-ring generated by $\{R, S, T\}$. Then $\mathcal{B}_3$ has 49 elements, and is generated as a semi-ring by R, S and T, and the relations*

$$RRR = R, \quad SSS = S, \quad TTT = T,$$
$$RR = SS = TT, \quad RS = ST = TR, \quad SR = TS = RT,$$
$$R \vee R = R, \quad S \vee S = S, \quad T \vee T = T,$$
$$R \vee S \vee T = R(R \vee S \vee T) = S(R \vee S \vee T) = T(R \vee S \vee T).$$

So it is a reduced borromean object in the category of semi-rings.

In order to prove this proposition (stated less precisely and without proof in [11] p. 147), we check the given relations, and we introduce U, C^+ and C^- by:

$$U := RR = SS = TT, \quad C^+ := RS = ST = TR, \quad C^- := SR = TS = RT.$$

Then we code the elements of $\mathrm{Mat}_3(\mathbb{B}_{\mathrm{Alg}})$ by $(m_{i,j}) \mapsto \{3(i-1)+j; m_{i,j} = 1\}$.

(a) So the six given elements are:

$$R = \{1,6,8\}, \quad S = \{3,5,7\}, \quad T = \{2,4,9\},$$
$$U = \{1,5,9\}, \quad C^+ = \{3,4,8\}, \quad C^- = \{2,6,7\},$$

and we see that (R, S, T) and (U, C^+, C^-) are two tripartitions of the set $\{1,2,3,4,5,6,7,8,9\}$.

Then the zero element in $\mathcal{B}_3$ is $0 = \{\}$, and we get the 42 other non-zero elements by the (non-empty) unions of generators:

(b) The 15 unique unions of 2 distinct generators:
– (9 of cardinality 5):
$\{1,2,4,5,9\}$, $\{1,2,6,7,8\}$, $\{1,3,4,6,8\}$, $\{1,3,5,7,9\}$, $\{1,5,6,8,9\}$, $\{2,3,4,8,9\}$, $\{2,3,5,6,7\}$, $\{2,4,6,7,9\}$, $\{3,4,5,7,8\}$;
– (6 of cardinality 6):
$\{1,2,4,6,8,9\}$, $\{1,2,5,6,7,9\}$, $\{1,3,4,5,8,9\}$, $\{1,3,5,6,7,8\}$, $\{2,3,4,5,7,9\}$, $\{2,3,4,6,7,8\}$.

(c) The maximal element is $\{1,2,3,4,5,6,7,8,9\}$ and it is the union of 3 distinct generators; it could be obtained so in two ways, as $R \vee S \vee T$ and as $U \vee C^- \vee C^+$.

(d) The 18 elements of cardinality 7 unique unions of 3 distinct generators, which are complements of elements of cardinal 2: $\{6,8\}'$, $\{6,7\}'$, $\{5,9\}'$, $\{5,7\}'$, $\{4,9\}'$, $\{3,8\}'$, $\{4,8\}'$, $\{3,7\}'$, $\{3,5\}'$, $\{3,4\}'$, $\{2,9\}'$, $\{2,7\}'$, $\{2,6\}'$, $\{2,4\}'$, $\{1,9\}'$, $\{1,8\}'$, $\{1,6\}'$, $\{1,5\}'$.

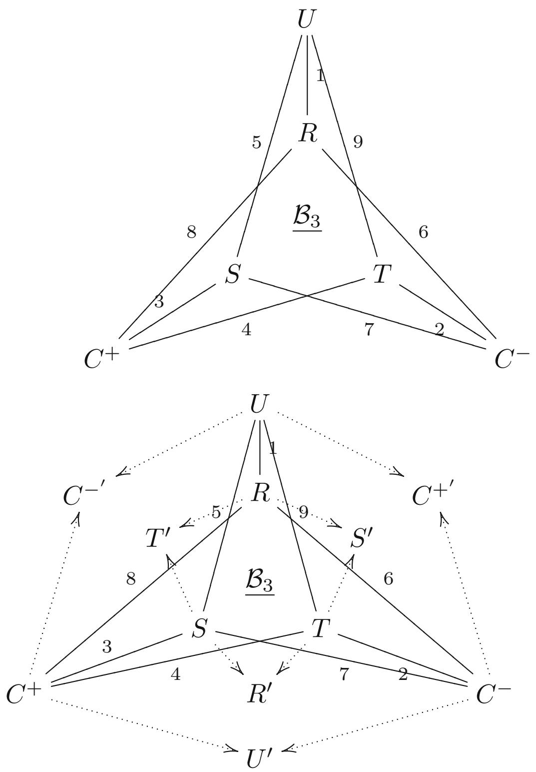
(e) The 9 elements of cardinality 8 unique unions of 4 distinct generators: they are the complements of singletons: $\{1\}'$, $\{2\}'$, $\{3\}'$, $\{4\}'$, $\{5\}'$, $\{6\}'$, $\{7\}'$, $\{8\}'$, $\{9\}'$.

In fact, starting from the six elements R, S, T, U, C^+, C^-, we observed that all the possible unions are different, with the exception of those with at least the three elements R, S, T or the three elements U, C^+, C^-, which are all equal to the same element 1. So $\mathcal{B}_3$ is generated by unions by these six elements and the only relation

$$R \vee S \vee T = U \vee C^- \vee C^+.$$

And to get the result it is enough to observe that $\mathcal{B}_3$ is stable by multiplication, as a consequence of the relation in the proposition and of the distributivity of the semi-ring. It is borromean in the sense that the generating relations are cyclic and if we add $R = U$ then $S = T = U$.

Proposition 4.2. *For the semi-ring $\mathcal{B}_3$ in Proposition 4.1, $\mathcal{H}\{R, S, T\}$ and $\mathcal{H}\{U, C^-, C^+\}$ the hexagon associated by Proposition 3.2 to the two 3-partitions $\{R, S, T\}$ and $\{U, C^-, C^+\}$, are related by the picture $\underline{\mathcal{B}_3}$:*



This "hexagonal" graph is a Moebius glueing of the three rectangles USC^+R, SC^+TC^-, TC^-RU, with frontier the circle USC^-RC^+TU.

In fact we have a magic square

	C^-	U	C^+
R	6	1	8
S	7	5	3
T	2	9	4

and each number $j \in \{1,2,3,4,5,6,7,8,9\}$ belongs exactly to the code of two of these six elements, one in the top row, the other in the left column.

5. Paradoxes, Squarings and Meanings in $\mathbb{F}_4$

Of course $\mathbb{F}_4$ is a boolean algebra. But considering it as an imaginary extension of the boolean algebra $\mathbb{F}_2$, it could be used as a tool for the analysis of paradoxes (paradoxical differences). So this calculus becomes a tool for a *theory of meanings* of paradoxical and impossible sentences.

Our point of view is that a *meaning* of a sentence is something in opposition to its logical value, it comes as a movement of thinking 'out of logic'. Especially a sentence could be both an antilogy *and* very meaningful: its main meaning is the way in which it is an antilogy. A *mathematical hermeneutics* would be a calculus of such movements, which are to be localized in the sentence by indications or signs of modification of logical operators. In the context of $\mathbb{F}_4$ it is possible to construct several games of such indications. We shall explain here the squaring and the construction of paradoxical differences. In the two next sections we will meet the direct indication of changing of classical boolean logic, and in the following section the calculus of speculations.

Proposition 5.1. 1. *Let* $\mathbb{Z}/2\mathbb{Z} = (\{0,1\},+,\times) = \mathbb{F}_2$ *be the field of integers modulo* 2, *and* $P = X^2+X+1 \in \mathbb{Z}/2\mathbb{Z}[X]$. *There is a smallest field—the Galois' field* $\mathbb{F}_4$*—in which* P *splits, with* 4 *elements:* $0, 1, \alpha, \omega$.
In fact on a set with 4 *elements as* $\mathbb{F}_4$ *there is up to isomorphism a unique structure of field, and there are exactly* 12 *isomorphic field structures, each one being determined by the choice of the* zero *element* Z *and the choice of* unity U, $U \neq Z$. *Here we limit our interest to just one given structure of field denoted by* $\mathbb{F}_4$.

2. *On the one hand* $\mathbb{F}_4$ *is a classical boolean algebra in itself, with a structure as* $(\neg_\kappa, \wedge_\kappa, \Rightarrow_\kappa)$, *and on the other hand this field could be used as an imaginary extension of the classical boolean field* $\mathbb{F}_2$ *as a field, it is then a splitting field of the logical paradox that there is no proposition* p *such that*

$$p = (\neg p \Leftarrow p)$$

In [6, pp. 64–66] we explained how, when we want to express the constitution of cartesian evidence as well as of freudian Unheimliche, we meet such a paradoxical object as a proposition p such that $p = (\neg p \Leftarrow p)$. This condition, in classical propositional calculus, where $p \Rightarrow q = pq + p + 1$ and $\neg p = p + 1$, is equivalent to $p = p(p+1) + p + 1, = p^2 + 2p + 1$. And as $2p = 0$ we get

$p^2 + p + 1 = 0$. Furthermore as $p^2 = p$, we get $p = p + 1$, and $0 = 1$. This proves that there are not classical real propositions with $p = (\neg p \Leftarrow p)$.

However if we imagine an extended calculus of *complex propositions* x for which $x^2 = x$ is not automatic, except for the "real" elements, then we could find some imaginary solution for the equation $X^2 + X + 1 = 0$ i.e. $P(X) = 0$. And precisely the splitting field $\mathbb{F}_4$ of P is the field with 4 elements, it is the set $\{0, 1, \alpha, \omega\}$, with 0 the unit for $+$, 1 the unit for $\times$, with α and ω the two roots of P, and so with addition and multiplication given by: $\alpha + \omega = 1$, $\alpha\omega = 1$. Every element x in this field satisfies $x^4 = x$, and the elements 0 and 1 are those satisfying $x^2 = x$. The nice idea of using $\mathbb{F}_4$ as a *splitting field of paradoxes* was introduced in 1973 by Grosjean [4].

Fixing in the linear space $\mathbb{F}_4$ the $\mathbb{F}_2$-basis $(\alpha, \omega) = \kappa$, every element is written as $x = a\alpha + z\omega$, with $a, z \in \mathbb{F}_2$. Then a classical boolean logic $(\neg_\kappa, \wedge_\kappa, \Rightarrow_\kappa)$ on $\mathbb{F}_4$ is given componentwise:

$$\neg_\kappa x = (\neg a)\alpha + (\neg z)\omega, x \wedge_\kappa x' = (a \wedge a')\alpha + (z \wedge z')\omega,$$
$$x \Rightarrow_\kappa x' = (a \Rightarrow a')\alpha + (z \Rightarrow z')\omega.$$

Of course, in this boolean structure the paradox always exists. But, if we know that for real elements $(x^2 = x, y^2 = y)$ the conjunction is expressed by the product in the field, $x \wedge_\kappa y = xy$, and the implication is expressed by $x \Rightarrow_\kappa y = xy + x + 1$, then the idea is to look at the expression

$$x = x(x + 1) + x + 1$$

for non-real elements, and then we get imaginary solutions for the paradox so expressed.

Now we pursue the job in a new way, towards an analysis of antilogies, and then towards an understanding of how the field structure itself is generated starting from *several* classical logics on $\mathbb{F}_4$.

Proposition 5.2. 1. *In fact the boolean logical structure* $(\neg_\kappa, \wedge_\kappa, \Rightarrow_\kappa)$ *of* $\mathbb{F}_4$, *where 'false'* $= f = 0$, *'true'* $= t = 1$, *is also expressible in algebraic form* (*with the field structure of* $\mathbb{F}_4$) *by:*

$$\neg_\kappa x = x + 1, x \wedge_\kappa y = x^2y^2 + x^2y + xy^2,$$
$$x \Rightarrow_\kappa y = x^2y^2 + x^2y + xy^2 + x + 1,$$

which of course, if $x^2 = x$ *and* $y^2 = y$ *reduces to*

$$\neg_\kappa x = x + 1, x \wedge_\kappa y = xy, x \Rightarrow_\kappa y = xy + x + 1.$$

So the classical logic of $\mathbb{F}_2$ *appears as a quotient of the classical logic of* $\mathbb{F}_4$ *by the identification* (*boolean reduction*)

$$x^2 \equiv x.$$

2. *Furthermore, to any function* P *on* $\mathbb{F}_4$*—logical or not logical—we could associate a boolean reduction* ${}^bP$ (*which is a logical function on* $\mathbb{F}_2$ *if the coefficients of* P *are in* $\mathbb{F}_2$), *and then* ${}^dP = P - {}^bP$ *could be interpreted as the* paradoxical difference *inherent to* P, *which is an* avatar of antilogy. *And the point is that such an avatar, though antilogical at the level of* $\mathbb{F}_2$, *could possibly get solutions in* $\mathbb{F}_4$.

Starting from the fact that any function on $\mathbb{F}_4$ could be expressed as a polynomial, by the method of undetermined coefficients these formulas for $\neg_\kappa$, $\wedge_\kappa$ and $\Rightarrow_\kappa$ could be obtained easily. But also we could directly write

$$P(x,y) = \Sigma_{(u,v)\in\mathbb{F}_4^2} P(u,v)[1-(x-u)^3][1-(y-v)^3] \qquad (\star)$$

As $x \wedge_\kappa x' = (a\alpha + z\omega) \wedge_\kappa (a'\alpha + z'\omega) = aa'\alpha + zz'\omega$, the formula for the conjunction comes from $x^2x' + xx'^2 = az' + a'z$ and $x^2x'^2 = aa'\alpha + zz'\omega + az' + a'z$. The function $c(x,y) = x \wedge_\kappa y - xy = x^2y^2 + x^2y + xy^2 - xy = xy(xy + x + y + 1)$ is 0 on real elements, but not on α and ω, where its values are 1; we named it an *avatar of antilogy.* Such an avatar is an incarnation or a presentation of an impossibility.

More systematically, as $\mathbb{F}_4$ is a finite field, and as $x^4 = x$, any function in two variables could be presented as a polynomial of degree ≤ 6 with coefficients in $\mathbb{F}_4$:

$$P = \sum_{i,j=0,\ldots,3} a_{i,j} x^i y^j.$$

Let ${}^bP$ be its *boolean logical reduction*, obtained by reducing x^2 to x and y^2 to y, i.e.

$${}^bP = \left(\sum_{i,j\geq 1} a_{i,j}\right) xy + \left(\sum_{i\geq 1, j=0} a_{i,j}\right) x + \left(\sum_{i=0, j\geq 1} a_{i,j}\right) y + a_{0,0}.$$

Then ${}^dP = P - {}^bP$ is its *paradoxical difference.*

Proposition 5.3. 1. *Starting with a boolean expression E with coefficients in $\mathbb{F}_2$, we can construct an avatar of E, i.e. an expression ${}^aE$ with values in $\mathbb{F}_4$ just by adding at will in various places some squaring operations $(-)^2$, and then ${}^{ba}E = E$, and ${}^aE - E$ is an avatar of antilogy. We call ${}^aE$ a* squaring avatar *or a* squarification *of E.*

2. *In the case of the implication $x \Rightarrow_\kappa y = x^2y^2 + x + (x^2y + xy^2 + 1)$, by adding squaring as in item 1 above we get four different squarifications:*

$$\begin{aligned} x \Rightarrow_\kappa y &= x^2y^2 + x + (x^2y + xy^2 + 1),\\ x^2 \Rightarrow_\kappa y^2 &= xy + x^2 + (xy^2 + x^2y + 1),\\ x^2 \Rightarrow_\kappa y &= xy^2 + x^2 + (xy + x^2y^2 + 1),\\ x \Rightarrow_\kappa y^2 &= x^2y + x + (xy + x^2y^2 + 1). \end{aligned}$$

Of these 'squarified' implications, only the first one is really boolean on all $\mathbb{F}_4$, i.e. the implication of one of the twelve isomorphic boolean structures on $\mathbb{F}_4$ (describe by Propositions 6.5 and 6.6).

Moreover, once introduced the description of the various boolean structures on $\mathbb{F}_4$, in the next section, we could see also in Proposition 9.2 how a squarified implication for a given boolean structure could be another squarified implication for others boolean structures.

6. Hexagonal Framework of $\mathbb{F}_4$

It would be possible to construct a systematic theory of *logical manifolds* (as geometrical glueings of boolean algebras, given by atlases of charts according to geometrical shapes), but here we only put forward the logic of $\mathbb{F}_4$ as an example.

Here we show how $\mathbb{F}_4$ could be organized as a logical manifold on a hexagonal framework (starting with the pointed borromean object in Proposition 3.5), and how this allows the introduction of logical differentials. Then the *theory of meanings* (cf. Proposition 8.2) will be really a consequence of this hexagonal manifold structure. In fact all that could be understood as a generating fragment of a symmetric system of 12 classical logics with movements and alterations in this system (Propositions 6.5 and 8.3).

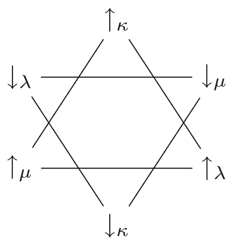
Proposition 6.1. *The $\mathbb{F}_2$-vector space $\mathbb{F}_4$ has 3 unordered bases: $\kappa = (\alpha, \omega)$, $\lambda = (1, \alpha)$, $\mu = (1, \omega)$, and if $\beta = (u, v)$ is one of these bases we take $t_\beta := u+v$, and we construct the pointwise boolean algebra $\mathbb{F}_4^\beta = (\neg_\beta, \wedge_\beta, \Rightarrow_\beta)$ of $\mathbb{F}_4$, where $0 =: f$, $t_\beta =: t$—and so $t^3 = 1$—with*

$$\neg_\beta(x) = x + t_\beta, \quad x \wedge_\beta y = x^2y^2 + t_\beta(x^2y + xy^2);$$

then $x \vee_\beta y = x^2y^2 + x + y + t_\beta(x^2y + xy^2)$, $x \Rightarrow_\beta y = x^2y^2 + x + t_\beta(x^2y + xy^2 + 1)$, and $x \Leftrightarrow_\beta y = x + y + t_\beta$. Of course these and all logical operators could be recovered from the Sheffer stroke $NOR_\beta(x, y) = \neg_\beta(x \vee_\beta y) = x \downarrow_\beta y$ as well as from its dual: $NAND_\beta(x, y) = \neg_\beta(x \wedge_\beta y) = x \uparrow_\beta y$

$$x \downarrow_\beta y = x^2y^2 + x + y + t_\beta(x^2y + xy^2 + 1) \quad x \uparrow_\beta y = x^2y^2 + t_\beta(x^2y + xy^2 + 1).$$

All that is encysted in the hexagonal data $x \uparrow_\beta y$ and $x \downarrow_\beta y$:



If we look for a binary logical operation which generates all the logical functions, Sheffer [18, p. 486] gave the first case $x|y = \neg x \wedge \neg y$ (what we note here $x \downarrow y$, or the NOR, or the *joint denial*), and Zylinski [22] proved that this example and its dual $x \uparrow y$ are the only cases.

The formulas in the case of λ and μ are proved as in the case of κ, by the method of undetermined coefficients or by $(\star)$ (in the proof of Proposition 5.2).

Proposition 6.2. *What we get on $\mathbb{F}_4$ in Proposition 6.1 is a* moving boolean logic*, with a parameter $t \in \{\alpha, \omega, 1\}$ moving on the hexagonal data of the $\uparrow$ and $\downarrow$, and with a calculus of* logical differentials *depending on the function*

$$\sigma(x, y) = x^2y + xy^2$$

and given by:

$$d(\Leftrightarrow_\beta) = d(\neg_\beta) = d(t_\beta), \quad d(\wedge_\beta) = d(\vee_\beta) = \sigma d(t_\beta),$$
$$d(\Rightarrow_\beta) = d(\uparrow_\beta) = d(\downarrow_\beta) = (\sigma + 1)d(t_\beta).$$

Comparing the three formulas for the negations and conjunctions we observe that in fact the only modification is expressible by introducing the t_β. In fact also with $x = a\alpha + z\omega$, $x' = a'\alpha + z'\omega$, we have $x \wedge_\kappa x' = aa'\alpha + zz'\omega$ and

$$x \wedge_\lambda x' = (a + z)(a' + z')\alpha + zz', \quad x \wedge_\mu x' = aa' + (a + z)(a' + z')\omega,$$

and, as it is easy to check $x^2x' + xx'^2 = az' + a'z$, the so called *logical differentials* for the conjunctions are:

$$x \wedge_\lambda x' - x \wedge_\kappa x' = (az' + a'z)\alpha = (x^2x' + xx'^2)(t_\lambda - t_\kappa),$$
$$x \wedge_\mu x' - x \wedge_\kappa x' = (az' + a'z)\omega = (x^2x' + xx'^2)(t_\mu - t_\kappa),$$
$$x \wedge_\lambda x' - x \wedge_\mu x' = (az' + a'z)1 = (x^2x' + xx'^2)(t_\lambda - t_\mu),$$

and this could be written $d(\wedge_\beta) = \sigma d(t_\beta)$.

Other connectors are given for example by $x \Rightarrow_\beta y = \neg_\beta(x \wedge_\beta \neg_\beta y)$ and so on, and the announced differentials are evident.

Proposition 6.3. *A map $f : \mathbb{F}_4 \to \mathbb{F}_4$ is $\mathbb{F}_2$-linear if and only if it is of the type $f(x) = ax^2 + bx$, and it is invertible in 6 cases (either $a = 0$ and $b \neq 0$, or $a \neq 0$ and $b = 0$):*

$$\mathrm{id}(x) = x, \quad \alpha(x) = \alpha x, \quad \omega(x) = \omega x,$$
$$1^\dagger(x) = \mathrm{sq}(x) = x^2, \quad \alpha^\dagger(x) = \alpha^2 x^2, \quad \omega^\dagger(x) = \omega^2 x^2.$$

So $\alpha = \alpha^\dagger \circ \omega^\dagger = \omega^\dagger \circ 1^\dagger = 1^\dagger \circ \alpha^\dagger$, $\omega = \omega^\dagger \circ \alpha^\dagger = \alpha^\dagger \circ 1^\dagger = 1^\dagger \circ \omega^\dagger$, and $\mathrm{GL}_2(\mathbb{F}_2) \simeq S(3)$.

Clearly x^2 and x are linear, and x^3 is not. The map f is invertible if and only if $a^3 + b^3 \neq 0$, and the inverse is

$$f^{-1}(x) = \frac{a}{a^3 + b^3}x^2 + \frac{b^2}{a^3 + b^3}x.$$

In fact $a^3 + b^3 \neq 0$ is equivalent to $a = 0$ and $b \neq 0$, or $a \neq 0$ and $b = 0$.

Proposition 6.4. *The 3 boolean structures $\mathbb{F}_4^\kappa, \mathbb{F}_4^\lambda, \mathbb{F}_4^\mu$ on the set $\{0, \alpha, \omega, 1\}$ are, among the 12 possible boolean structures (see Proposition 6.5), those for which the addition (or symmetric difference) is the one fixed as the addition '+' in $\mathbb{F}_4$, those for which the false is $f = 0$. We call them the* basic boolean structures *on $\mathbb{F}_4$.*

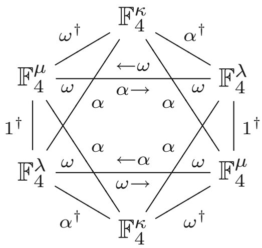
Given a basis $\beta = (u, v)$ of $\mathbb{F}_4$, any linear invertible map $f : \mathbb{F}_4 \to \mathbb{F}_4$ carries isomorphically the boolean logic $\mathbb{F}_4^\beta$ towards $\mathbb{F}_4^{f\beta}$ with $f\beta = (f(u), f(v))$, by 'moving frame' formulas:

$$\neg_{f\beta} x = f(\neg_\beta f^{-1}(x)), \quad x \wedge_{f\beta} y = f(f^{-1}(x) \wedge_\beta f^{-1}(y)).$$

And so we get an action of $\mathrm{GL}_2(\mathbb{F}_2) \simeq \mathcal{S}(3)$ on the set of basic boolean structures $\{\mathbb{F}_4^\kappa, \mathbb{F}_4^\lambda, \mathbb{F}_4^\mu\}$ given in abridged notations by

$$\mathrm{id} = \mathrm{id}, \quad 1^\dagger = (\lambda, \mu), \quad \omega^\dagger = (\mu, \kappa), \quad \alpha^\dagger = (\kappa, \lambda), \quad \alpha = (\kappa, \mu, \lambda), \quad \omega = (\lambda, \mu, \kappa).$$

These 6 *transformations are* illogical, *they* are not *defined as logical functions, although they define homomorphisms of logical structures*; *here we will call them the* (illogical) transition maps *or* change of (logical) charts, *because we can understand* $\mathbb{F}_4$ *as a field by patching together the boolean algebras* $\mathbb{F}_4^\beta$ *along these transformations, and so as a* logical manifold, *with a hexagonal atlas.*



We think of $\mathbb{F}_4$ *as an object in the middle of this hexagon, a '*boolean manifold*' with three initial charts defined by the identity function:* $r : \mathbb{F}_4^\kappa \to \mathbb{F}_4$, $s : \mathbb{F}_4^\lambda \to \mathbb{F}_4$, $t : \mathbb{F}_4^\mu \to \mathbb{F}_4$*: by these maps the three families of logical functions are brought together and could be mixed and composed, and composed also with the change of charts; of course the compositions are no more logical.*

Proposition 6.5. 1. *On* $\mathbb{F}_4$ *there are* 24 *bijections* (*constituting* $S(4)$), *given by* $b_{p,1,d}(x) = px + d$ *and by* $b_{p,2,d}(x) = px^2 + d$, *with* $p \neq 0$ (*i.e.* $p^3 = 1$), *with inverses* $b^{-1}_{p,1,d}(x') = p^2x' + p^2d$ *and* $b^{-1}_{p,2,d}(x') = px'^2 + pd^2$. *These bijections are the* $\mathbb{F}_2$*-affine maps translations of the 6 linear maps given in Proposition* 6.3. *So* $\mathcal{S}(4) \equiv \mathrm{GA}_2(\mathbb{F}_2)$.

2. *If we transfer the boolean structure* $\mathbb{F}_4^\kappa$ *by the* 24 *bijections, we get on* $\mathbb{F}_4$ *the* 12 *different isomorphic boolean logical structures* $\mathbb{F}_4^\xi$, *each* $\xi = (f, t)$ *being given by the choice of 'false'* $= f$ *and the 'true'* $= t$, *with* $t \neq f$, *with explicitly*

$$\neg_\xi(x) = x + f + t,\ x \wedge_\xi y = x^2y^2 + (f + t)(x^2y + xy^2) + ft(x^2 + y^2) + (f + t)f^2(x + y),$$
$$x \vee_\xi y = x \wedge_\xi y + x + y, \quad x \Rightarrow_\xi y = x \wedge_\xi y + x + t,$$
$$NOR_\xi(x, y) = x \downarrow_\xi y = x \wedge_\xi y + x + y + f + t,$$
$$NAND_\xi(x, y) = x \uparrow_\xi y = x \wedge_\xi y + f + t.$$

3. *The* boolean duality *is realized as*

$$\xi = (f, t), \quad \xi^{\mathrm{op}} = (t, f),$$

and then

$$\neg_{\xi^{\mathrm{op}}} = \neg_\xi, \quad \wedge_{\xi^{\mathrm{op}}} = \vee_\xi, \quad \vee_{\xi^{\mathrm{op}}} = \wedge_\xi.$$

4. *The datum of each boolean algebra* $\mathbb{F}_4^\xi = (\neg_\xi, \wedge_\xi, \Rightarrow_\xi)$ *is equivalent to a boolean ring* $(\mathbb{F}_4, \otimes_\xi, \oplus_\xi)$ *with*

$$x \otimes_\xi y = x \wedge_\xi y, \quad x \oplus_\xi y = x + y + f.$$

We transfer $\neg_\kappa = \neg_1$ and $\wedge_\kappa = \wedge_1$ to $\neg_\xi$ and $\wedge_\xi$, with $\xi = (f,t)$, via $\phi = b_{p,1,d}$, with $t = \phi^{-1}(1)$, $f = \phi^{-1}(0)$ by

$$\neg_\xi(x) = \phi^{-1}(\neg_1(\phi(x)), x \wedge_\xi y = \phi^{-1}(\phi(x) \wedge_1 \phi(y)).$$

At first we get $f = p^2 d$ and $p^2 + f = t$, and then

$$\phi(x) = (f+t)^2(x+f) = x', \quad \phi^{-1}(x') = (f+t)x' + f = x.$$

Then we compute $\neg_\xi(x) = (f+t)[(f+t)^2(x+f)+1] + f = x + f + t$, and

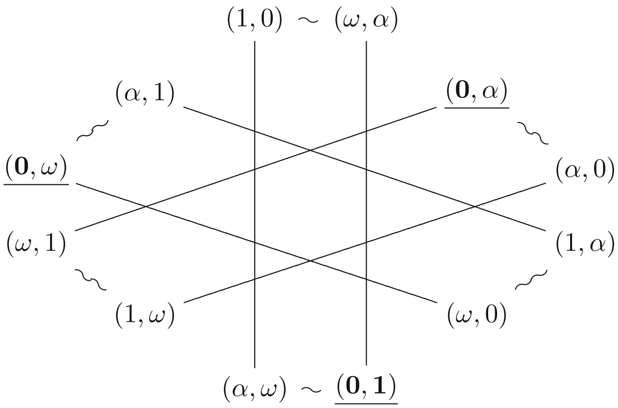
$$x \wedge_\xi y = (x+f)^2(y+f)^2 + (f+t)[(x+f)^2(y+f) + (x+f)(y+f)^2] + f,$$

which we expand as the announced formula.

From the case $\xi = (0,1)$, where we know that $x' \vee_1 y' = x' \wedge_1 y' + x' + y'$, $x' \Rightarrow_1 y' = x' \wedge_1 y' + x' + 1$, $x' \uparrow_1 y' = x \wedge_1 y + 1$, we get the last formulas, via ϕ.

For the duality it is clear, and finally we get $x \oplus_\xi y = (\neg_\xi \wedge_\xi y) \vee_\xi (x \wedge_\xi \neg_\xi y)$, that is to say $f + \neg_\xi x \wedge_\xi y + x \wedge_\xi \neg_\xi y = x + y + f$.

Proposition 6.6. 1. *The* 12 *boolean structures on* $\mathbb{F}_4 = \{0, \alpha, \omega, 1\}$ *expressed in Proposition* 6.5 *could be arrange in a bi-hexagon*



following the values of $\xi = (f,t)$, *where the central symmetry is* $((f,t) \mapsto (f+1, t+1))$, *the ondulating lines are* $(f,t) \mapsto (f+\alpha, t+\alpha)$ *and the straight lines are* $(f,t) \mapsto (f+\omega, t+\omega)$.

2. *These three movements are the three oppositions in a square like the Aristoteles–Apuleius square* $\mathcal{AA}$ (*Proposition* 2.3); *each of these oppositions could be seen also as the negation at work for each of the boolean structures at the vertices of one of the three quadrilaterals (squares). So in the four logics of the quadrilateral* $(0,1), (\alpha,\omega), (1,0), (\omega,\alpha)$ *the negation is* $(-)+1$, *in the four logics of the quadrilateral* $(0,\alpha), (\alpha,0), (1,\omega), (\omega,1)$ *the negation is* $(-)+\alpha$, *in the four logics of the quadrilateral* $(0,\omega), (\alpha,1), (1,\alpha), (\omega,0)$ *the negation is* $(-)+\omega$.

3. *So this picture exhibits the system of the* 12 *classical boolean logics on a set of cardinal* 4 *as a kind of 'borromean link' of three negations, with, for each negation, a 'square' of* 4 *possible conjunctions; it is a planar alternated projection like the second borromean link in the Tait's series* [19]

(*after the case of the first in Proposition* 2.8). *We call it* bi-hexagonal borromean link.

4. *The glueing* $(0,1) \equiv (\omega,0)$, $(1,\alpha) \equiv (\alpha,0)$, $(0,\alpha) \equiv (\omega,\alpha)$, $(1,0) \equiv (\alpha,1)$, $(0,\omega) \equiv (\omega,1)$ *and* $(1,\omega) \equiv (\alpha,\omega)$, *generates the octahedral hexagon* (*Propositions* 2.4 *and* 2.5) *as a quotient of the bi-hexagon.*

7. The 6 Boolean Logic Functions Algebras on $\mathbb{F}_4$

The hexagonal framework inherent to $\mathbb{F}_4$ exhibited in the previous section is reenforced here by the presentation of the hexagon of the six full boolean logic functions sub-algebras of

$$\mathbb{P}_4 = \mathbb{P}(\mathbb{F}_4) = \bigcup_{n\geq 0} \mathbb{F}_4^{\mathbb{F}_4^n},$$

the Post-Mal'cev full iterative algebra of functions of all arities on $\mathbb{F}_4$ (defined in [13, pp. 30–31]).

Proposition 7.1. 1. *An arbitrary function* $F : \mathbb{F}_4^2 \to \mathbb{F}_4$ *is one of the twelve possible boolean conjunctions if and only there exist a* $A \in \{\alpha,\omega,1\}$ *and a* $B \in \{0,\alpha,\omega,1\}$ *such that, for every* $x, y \in \mathbb{F}_4$

$$\begin{aligned} F(x,y) &= x^2y^2 + A(x^2y + xy^2) + A^2(B^2+B)(x^2+y^2) + B(x+y) \\ &:= \&_{[A,B]}(x,y), \end{aligned}$$

and then $F = \wedge_\xi$ *with* $\xi = (f,t)$,

$$\begin{aligned} f = AB^2, &\quad t = AB^2 + A, \\ A = f + t &\quad B = (f+t)f^2. \end{aligned}$$

Especially if $\&_{[A,B]} = \wedge_\xi$, *then* $\&_{[A,B+1]} = \wedge_{\xi^{\mathrm{op}}} = \vee_\xi$, *and so* $B \mapsto B+1$ *realizes the boolean duality.*

2. *Given* F *and* G *two such conjunctions, the function* $\Delta = G - F$ *will be called a* logical differential of conjunction, *and the study of the system of Proposition* 6.6 *will be equivalent to a calculus of such logical differentials associated to movements in the picture* (*extending the calculus in Proposition* 6.1 *and Proposition* 6.2).

If $a \neq b$ and $\{a,b\} = \{f,t\}$, the data of $a+b$ and ab determine $\{a,b\}$, but do not permit to say which element in $\{a,b\}$ is the false f and which is t, in such way to make the distinction between ξ and ξ^{op}, between $\wedge_\xi$ and $\vee_\xi$. So the parametrization by (A,B) is better than the parametrization by $f+t$ and ft, and is really equivalent to the datum of ξ.

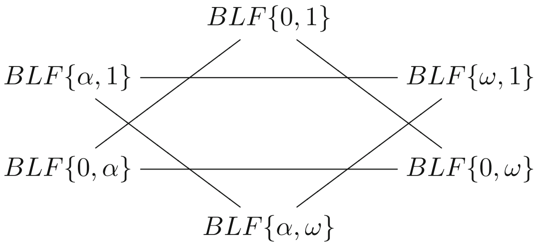
We consider $\mathbb{P}_4 = \mathbb{P}(\mathbb{F}_4) = \bigcup_{n\geq 0} \mathbb{F}_4^{\mathbb{F}_4^n}$. For an arbitrary function $\phi : \mathbb{F}_4^n \to \mathbb{F}_4$, we denote by $[\phi]$ the full sub-algebra of $\mathbb{P}_4$ generated by ϕ, and we say that ϕ is a *boolean logic generator* if $[\phi]$ is isomorphic to a *full algebra of boolean logic functions*, i.e. to $\mathbb{P}_2 = \mathbb{P}(\mathbb{F}_2)$.

In fact if W is a full algebra of boolean logic functions, we know that there are exactly two boolean logic generators of arity 2, the NOR and the NAND of Sheffer [18,22]. So with 6.5 and 7.1, we get the

Proposition 7.2. 1. *In $\mathbb{P}(\mathbb{F}_4)$ there are exactly twelve binary boolean logic generators which are, with $A \in \{\alpha, \omega, 1\}$ and $B \in \{0, \alpha, \omega, 1\}$:*

$$S_{[A,B]}(x,y) := x^2y^2 + A(x^2y + xy^2 + 1) \\ +A^2(B^2 + B)(x^2 + y^2) + B(x + y).$$

2. *In $\mathbb{P}(\mathbb{F}_4)$ there are exactly six full algebras of boolean logic functions, and each one has two binary boolean logic generators $S_{[A,B]}$ and $S_{[A,B+1]}$, and is denoted by $[S_{[A,B]}] = [S_{[A,B+1]}] =: BLF\{f,t\}$. These algebras are organized in an hexagon*



8. The 12 Speculations in $\mathbb{F}_4$

In the context of $\mathbb{F}_4$ and its boolean structure $\mathbb{F}_4^{\kappa}$, we construct an interpretation of modified logic [20] and of specular logic [5]. Then we extend this construction to the twelve boolean structures of $\mathbb{F}_4$ exhibited in the previous section, and this generates exactly 12 speculations.

Often in a natural dialogue when one says "x", he means in fact something like "x under the condition u", where u is obvious for him and not expressed, and u is to be discovered or speculated by the interlocutor, to get "the" correct meaning. Moreover, more precisely, "x under the condition u" could means $x \wedge u =: x^{\flat u}$ (*projection*) or $u \Rightarrow x =: x^{\sharp u}$ (*ejection*) in such a way that

$$x^{\flat u} = x \wedge u \leq x \leq u \Rightarrow x = x^{\sharp u}.$$

This makes sense if x is seen as a subset of a set E, u as a subset of E, or if x and u are elements of any boolean algebra, for example elements of $\mathbb{F}_4^{\xi}$.

The method of specular logic is introduced in [5], applied to psychoanalysis and to linguistics elsewhere, and here we just give as a quick example the freudian situation of 'denegation', as in the sentence

S = "It is my mother and it is not my mother".

The logical support A of S looks like $A = X \wedge \neg X$, and it is an antilogy. If we imagine that the first X is from the point of view $\flat\alpha$ and that the second is from the point of view $\sharp\omega$, then A is replaced by $Q = X^{\flat\alpha} \wedge_{\kappa} \neg_{\kappa} X^{\sharp\omega}$, which of course is not constant, and so is meaningful: it is one of the possible meanings of S.

In Sect. 5 we introduced the "squaring" as another way to obtain speculations for the construction of meanings, and in Sect. 6 the indication of moving among the 12 boolean structures on $\mathbb{F}_4$ could play the same part.

These three calculus (squaring, moving, speculation) could be used simultaneously, for instant by writing on $A = X \wedge \neg X$ indications as for example $(X^2)^{\flat\alpha} \wedge_\kappa \neg_\xi (X^{\sharp\omega})^2$.

Now we will give a precise description of the system of speculations in $\mathbb{F}_4$.

Proposition 8.1. 1. *Computing in $\mathbb{F}_4$ as a field and with the basis κ and its associated logic, we can realize the 'Specular Logic' of* [5] *with κ_α and κ_ω as points of view. So we can introduce four* speculations *for any element $x = a\alpha + z\omega$ by*

$$x^{\flat\kappa_\alpha} = a\alpha = x \wedge_\kappa \alpha, x^{\sharp\kappa_\alpha} = a\alpha + \omega = \alpha \Rightarrow_\kappa x,$$
$$x^{\flat\kappa_\omega} = z\omega = x \wedge_\kappa \omega, x^{\sharp\kappa_\omega} = \alpha + z\omega = \omega \Rightarrow_\kappa x.$$

2. *In $\mathbb{F}_4$ the construction of 'Modified Logic' of Vappereau* [20] *is possible, introducing* assertion *of $c \in \mathbb{F}_4$ and* complexification *of $r \in \mathbb{F}_2$ by the formulas $\vdash c := c + \omega$ and $r^\circ := r + \omega$, in such a way that $\vdash r^\circ = r$. The* modified negation *is then: $\sim x = (\neg_\kappa x) \wedge_\kappa \alpha = (\neg_\kappa x)^{\flat\kappa_\alpha}$.*

Proposition 8.2. *Starting with a sentence S of which the logical support seems 'to be' an antilogy A, we could construct a* meaning *of S by adding suitable speculations on A, in order to get a non-constant function Q on $\mathbb{F}_4$ such that the erasure of speculations gives back A.*

Proposition 8.3. 1. *The boolean algebra $\mathbb{F}_4^\beta$ (cf. Proposition 6.1) is equipped with four non-logical but linear or affine operations of* speculations *(two* projections *($\flat$) and two* ejections *($\sharp$)) given for any element $x = pu + qv$ as its minimal and maximal 'perceptions' limited to the 'points of view' u and v:*

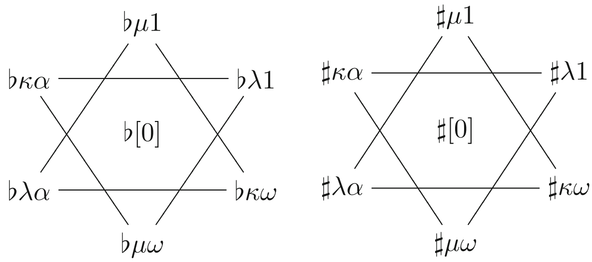
$$x^{\flat\beta_u} = pu = x \wedge_\beta u = uvx^2 + (1 + u^2v)x,$$
$$x^{\sharp\beta_u} = pu + v = u \Rightarrow_\beta x = uvx^2 + (1 + u^2v)x + v,$$
$$x^{\flat\beta_v} = qv = x \wedge_\beta v = uvx^2 + (1 + uv^2)x,$$
$$x^{\sharp\beta_v} = u + qv = v \Rightarrow_\beta x = uvx^2 + (1 + uv^2)x + u.$$

2. *These speculations could be obtained by adding elements of $\{1^\dagger, \alpha^\dagger, \omega^\dagger\}$ with elements of $\{1, \alpha, \omega\}$:*

$$(-)^{\flat\kappa\alpha} = 1^\dagger + \omega, \quad (-)^{\flat\kappa\omega} = 1^\dagger + \alpha, \quad (-)^{\flat\lambda 1} = \omega^\dagger + \omega,$$
$$(-)^{\flat\lambda\alpha} = \omega^\dagger + \alpha, \quad (-)^{\flat\mu 1} = \alpha^\dagger + \alpha, \quad (-)^{\flat\mu\omega} = \alpha^\dagger + \omega.$$

So they are determined by the transition maps and by the logical manifold structure.

3. *These speculations, relative to the boolean structures $\mathbb{F}_4^\beta$ for which the false is $f = 0$, are organized in two stars $\flat[0]$ and $\sharp[0]$:*



Proposition 8.4. *The speculations in Proposition 8.3 determine non-boolean* alterations *of the logic, by the introduction of non-boolean operations as*

$$x^{\flat\beta u} = x \wedge_\beta u, \quad x^{\sharp\beta u} = u \Rightarrow_\beta x,$$
$$\neg^{\flat\beta u} x = (\neg_\beta x) \wedge_\beta u, \quad \neg^{\sharp\beta u}(x) = u \Rightarrow_\beta (\neg_\beta x),$$
$$x \wedge^{\flat\beta u} y = (x \wedge_\beta y) \wedge_\beta u, \quad x \wedge^{\flat\beta u} y = u \Rightarrow_\beta (x \wedge_\beta y).$$
$$x \vee^{\flat\beta u} y = (x \vee_\beta y) \wedge_\beta u, \quad x \vee^{\flat\beta u} y = u \Rightarrow_\beta (x \vee_\beta y).$$
$$x \Rightarrow^{\flat\beta u} y = (x \Rightarrow_\beta y) \wedge_\beta u, \quad x \Rightarrow^{\flat\beta u} y = u \Rightarrow_\beta (x \Rightarrow_\beta y).$$

Proposition 8.5. *With the "$\xi = (f, t)$" notations, we identify κ, λ and μ with, respectively, $(0, 1)$, $(0, \omega)$ and $(0, \alpha)$, and the twelve speculations obtained in Proposition 8.3 are*:

$$x^{\flat\kappa\alpha} = x \wedge_{(0,1)} \alpha = x^2 + \omega x, \quad x^{\sharp\kappa\alpha} = \alpha \Rightarrow_{(0,1)} x = x^2 + \omega x + \omega,$$
$$x^{\flat\kappa\omega} = x \wedge_{(0,1)} \omega = x^2 + \alpha x, \quad x^{\sharp\kappa\omega} = \omega \Rightarrow_{(0,1)} x = x^2 + \alpha x + \alpha;$$

$$x^{\flat\lambda\alpha} = x \wedge_{(0,\omega)} \alpha = \alpha x^2 + \alpha x, \quad x^{\sharp\lambda\alpha} = \alpha \Rightarrow_{(0,\omega)} x = \alpha x^2 + \alpha x + 1,$$
$$x^{\flat\lambda 1} = x \wedge_{(0,\omega)} 1 = \alpha x^2 + \omega x, \quad x^{\sharp\lambda 1} = 1 \Rightarrow_{(0,\omega)} x = \alpha x^2 + \omega x + \alpha;$$

$$x^{\flat\mu 1} = x \wedge_{(0,\alpha)} 1 = \omega x^2 + \alpha x, \quad x^{\sharp\mu 1} = 1 \Rightarrow_{(0,\alpha)} x = \omega x^2 + \alpha x + \omega,$$
$$x^{\flat\mu\omega} = x \wedge_{(0,\alpha)} \omega = \omega x^2 + \omega x, \quad x^{\sharp\mu\omega} = \omega \Rightarrow_{(0,\alpha)} x = \omega x^2 + \omega x + 1.$$

Proposition 8.6. *Any affine map $x \mapsto ax^2 + bx + c$—and especially the squaring $x \mapsto x^2$—is a sum of speculations given in Proposition 8.5*:

$$x^{\flat\lambda\alpha} + x^{\flat\mu 1} = x^{\flat\lambda 1} + x^{\flat\mu\omega} = x^2,$$
$$x^{\flat\kappa\omega} + x^{\flat\mu 1} = \alpha x^2, x^{\flat\kappa\alpha} + x^{\flat\lambda 1} = \omega x^2,$$
$$x^{\flat\kappa\alpha} + x^{\flat\kappa\omega} = x^{\flat\lambda\alpha} + x^{\flat\lambda 1} = x^{\flat\mu 1} + x^{\flat\mu\omega} = x,$$
$$x^{\flat\lambda\alpha} + x^{\flat\mu 1} + x^{\flat\kappa\omega} = \alpha x, \quad x^{\flat\lambda\alpha} + x^{\flat\mu 1} + x^{\flat\kappa\alpha} = \omega x,$$
$$x^{\flat\kappa\omega} + x^{\sharp\kappa\omega} = x^{\flat\lambda 1} + x^{\sharp\lambda 1} = \alpha, \quad x^{\flat\kappa\alpha} + x^{\sharp\kappa\alpha} = x^{\flat\mu 1} + x^{\sharp\mu 1} = \omega,$$
$$x^{\flat\lambda\alpha} + x^{\sharp\lambda\alpha} = x^{\flat\mu\omega} + x^{\sharp\mu\omega} = 1.$$

Proposition 8.7. 1. *We get speculations for any of the twelve boolean logical structures ξ on $\mathbb{F}_4$ (as described in Propositions 6.5 and 6.6) given* *by* $(\neg_\xi, \wedge_\xi)$ *and for any* $u \notin \{f, t\}$. *All these speculations could be constructed with the only new operation of squaring* $(-)^2$. *For example any* projection

$$x^{\flat \xi u} = x \wedge_\xi u,$$

with $\xi = (f, t)$ *and* $u \notin \{f, t\}$, *is given as*

$$x \wedge_\xi u = [u^2 + (f+t)u + ft]x^2 + [(f+t)u^2 + (f+t)f^2]x + [ftu^2 + (f+t)f^2 u].$$

2. *If* u *and* v *are the atoms of the boolean structure* (f, t) *then*

$$u \Rightarrow_{(f,t)} x = x \wedge_{(t,f)} v,$$
$$x \wedge_{(f,t)} u = x \wedge_{(u,v)} f.$$

3. *So all the a priori* 48 *speculations are only* 12, *they all could be seen as projections as well as ejections, and their complete list is the one given in Proposition* 8.5. *These* 12 *speculations could be named the* 12 logical speculations, *and they are like modalities expressing* 12 'points of view'.

9. Borromean Structure of $\mathbb{P}_4 = \mathbb{P}(\mathbb{F}_4)$

In previous sections we saw that the hexagonal setting for $\mathbb{F}_4$ permits a moving boolean logic, made of logical differentials, and a calculus of meanings of paradoxes, by squaring and by paradoxical differences, or by speculations on 12 points of view. Now to conclude we will prove that any function on $\mathbb{F}_4$ can be produced by the compositions of the three boolean logical structures $\mathbb{F}^\kappa$, $\mathbb{F}^\lambda$, $\mathbb{F}^\mu$ indicated in the hexagon. This in fact specifies that the Post-Mal'cev *full iterative algebra* $\mathbb{P}_4 = \mathbb{P}(\mathbb{F}_4)$ of functions of all arities on $\mathbb{F}_4$ (see the beginning of Sect. 7), is a borromean mixture of three algebras of boolean functions, and is what we call the algebra of a *borromean logic*.

Proposition 9.1. *We have*

$$x^2 = x \wedge_\kappa 1 + x \wedge_\alpha 1 + x \wedge_\omega 1,$$
$$xy = x^2 \wedge_\kappa y + x \wedge_\kappa y^2 + x^2 \wedge_\kappa y^2,$$
$$\alpha x = \omega \wedge_\kappa x + x^2, \quad \omega x = \alpha \wedge_\kappa x + x^2;$$

and so for every n, *any function* $f : \mathbb{F}_4^n \to \mathbb{F}_4$ *could be expressed by a composition of* $\neg_\kappa$, $\wedge_\kappa$, *constant functions* $\alpha^!$ *and* $\omega^!$, *and* $(-)^2$, *and also could be expressed by a composition of* $\neg_\kappa$, $\wedge_\kappa$, $\neg_\alpha$, $\wedge_\alpha$, $\neg_\omega$, $\wedge_\omega$, *and constant functions* $\alpha^!$ *and* $\omega^!$.

So we get for any function $f(x, y)$ *a logical function* $l_f(x, y, \alpha, \omega)$ *of which* f *is a squaring avatar. And the same works for any function* $f(x_1, \ldots, x_n)$ *with an* $l_f(x_1, x_2, \ldots, x_n, \alpha, \omega)$.

So any function of the algebra $\mathbb{P}_4 = \mathbb{P}(\mathbb{F}_4) = \cup_{n \geq 0} \mathbb{F}_4^{\mathbb{F}_4^n}$ *(with the notation* $\mathbb{P}_4$ *similar to the one* P_4 *used by Lau in* [13], *in honor of Emil Post) is a squaring avatar of a logical function with two parameters.*

As $1+\alpha+\omega=0$, the first formula comes by adding:

$x \wedge_\kappa 1 = x^2 1^2 + 1(x^2 1 + x 1^2)$,
$x \wedge_\alpha 1 = x^2 1^2 + \alpha(x^2 1 + x 1^2)$,
$x \wedge_\omega 1 = x^2 1^2 + \omega(x^2 1 + x 1^2)$.

We could also use of formulas for x^2 in Proposition 8.5.

For the second formula we add:
$x^2 \wedge_\kappa y = xy^2 + (xy + x^2y^2)$,
$x \wedge_\kappa y^2 = x^2y + (x^2y^2 + xy)$,
$x^2 \wedge_\kappa y^2 = xy + (xy^2 + x^2y)$,
and the last two are special cases of the second one.

Also we have

$$x + y = (x \wedge_\kappa \neg_\kappa y) \vee_\kappa (\neg_\kappa x \wedge_\kappa y),$$

and we can conclude, as any function on the field $\mathbb{F}_4$ is polynomial.

It is also possible to proceed from the Theorem 1.4.2 (a) in [13, p. 100], which allows to generate any function on $\mathbb{F}_4$ with $\wedge_\kappa$, $\vee_\kappa$, the constant functions $\alpha^!(x)=\alpha$ and $\omega^!(x)=\omega$, and with the characteristic maps used in fact in the $(\star)$ decomposition (after the Proposition 5.2): $j_u(x) = \begin{cases} 1 & \text{if } x = u \\ 0 & \text{otherwise,} \end{cases}$

$$j_0(x) = 1-(x-0)^3 = x^3+1, \;\; j_1(x) = 1-(x-1)^3 = x^3+x^2+x,$$
$$j_\alpha(x)=1-(x-\alpha)^3 = x^3+\alpha x^2+\omega x, \;\; j_\omega(x)=1-(x-\omega)^3 = x^3+\omega x^2+\alpha x.$$

Proposition 9.2. *Starting with $x \Rightarrow_\beta y = x^2y^2 + x + t_\beta(x^2y + xy^2 + 1)$, for $t = 1, \alpha$ or ω, we abridge $x \Rightarrow_\beta y$ by $x \Rightarrow_t y$ with $t = t_\beta$; by adding squaring as in Proposition 5.3 we get the following squarifications (eight different if $t \neq 1$, four if $t = 1$):*

$$x \Rightarrow_t y, x^2 \Rightarrow_t y, x \Rightarrow_t y^2, x^2 \Rightarrow_t y^2,$$
$$(x \Rightarrow_t y)^2, (x^2 \Rightarrow_t y)^2, (x \Rightarrow_t y^2)^2, (x^2 \Rightarrow_t y^2)^2.$$

In fact the last four squares provide implications following other points of view:

$$(x \Rightarrow_t y)^2 = x^2 \Rightarrow_{t^2} y^2, \quad (x^2 \Rightarrow_t y)^2 = x \Rightarrow_{t^2} y^2,$$
$$(x \Rightarrow_t y^2)^2 = x^2 \Rightarrow_{t^2} y, \quad (x^2 \Rightarrow_t y^2)^2 = x \Rightarrow_{t^2} y.$$

So we get twelve squaring avatars of classical implications.

Proposition 9.3. *The multiplication in the field $\mathbb{F}_4$ is in several ways a kind of barycenter or mean value of conjunctions:*

$$xy = x^2 \wedge_\kappa y + x \wedge_\kappa y^2 + x^2 \wedge_\kappa y^2,$$
$$xy = x^2 \wedge_1 y^2 + x^2 \wedge_\alpha y^2 + x^2 \wedge_\omega y^2,$$
$$xy = (x \wedge_1 y)^2 + (x \wedge_\alpha y)^2 + (x \wedge_\omega y)^2,$$
$$xy = \Sigma_{(i,j,k)\in\{1,\alpha,\omega\}^3}(x \wedge_i 1) \wedge_j (1 \wedge_k y).$$

The first formula is the one given in Proposition 9.1, the second comes by adding $x^2 \wedge_t y^2 = xy + t(x^2y + xy^2)$, for $t = 1, \alpha, \omega$, the third comes from the first and the easy to check formulas

$$(x \wedge_1 y)^2 = x^2 \wedge_1 y^2, (x \wedge_\alpha y)^2 = x^2 \wedge_\omega y^2, (x \wedge_\omega y)^2 = x^2 \wedge_\alpha y^2,$$

and the fourth results of the second and the first in the Proposition 9.1.

Proposition 9.4. 1. *Let $\mathbb{F}_4$ be equipped with the totally linear order $\prec$ given by*

$$0 \prec \alpha \prec \omega \prec 1,$$

with the cyclic Post's negation $\sim_\prec (x)$ given by $\sim_\prec: 0 \mapsto \alpha \mapsto \omega \mapsto 1 \mapsto 0$, and with the conjunction $x \curlywedge y = \inf_\prec(x, y)$. Of course neither $\sim_\prec$ nor $\curlywedge$ are boolean operations. It is known [16, pp. 181–182] *that any element of $\mathbb{P}_4 = \mathbb{P}(\mathbb{F}_4)$ could be obtained by compositions of $\sim_\prec$ and $\curlywedge$. Furthermore Webb* [21] *introduced $x \Uparrow y = \sim_\prec (x \curlywedge y)$—as a generalized Sheffer's function—in such a way that $\sim_\prec (x) = x \Uparrow x$ and $x \curlywedge y = \sim^3_\prec (x \Uparrow y)$, and so any element of $\mathbb{P}_4 = \mathbb{P}(\mathbb{F}_4)$ could be obtained by composition of $\Uparrow$.*

2. *We have*

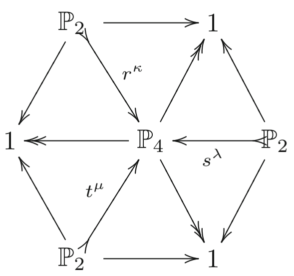
$$\sim_\prec (x) = \alpha + \alpha x^2,$$
$$x \curlywedge y = x^2y^2 + \alpha(x^3y^2 + x^2y^3 + x^3y + xy^3) + \omega(x^2y + xy^2),$$
$$x \Uparrow y = \alpha + \alpha xy + \omega(x^2y + xy^2) + (x^3y + xy^3) + (x^3y^2 + x^2y^3).$$

In fact we check with tables of values that

$$x \Uparrow y = \alpha + x^3y^3 + (x^2 + \alpha x)(y^2 + \alpha y) + (x^3 + x^2 + x)(y^3 + y^2 + y),$$

and then a simple expansion provides the last formula, and then the other formulas are easy to check.

Proposition 9.5. 1. *The algebra $\mathbb{P}_4 = \mathbb{P}(\mathbb{F}_4)$, which is generated by the binary operation $\Uparrow$ (Proposition 9.4), can be seen as generated by $\mathbb{P}_2 = P(\mathbb{F}_2)$ by construction of a squaring modality (see Proposition 9.1) over this boolean algebra. It is also the center of a picture in the manner of a reduced borromean object diagram:*



where r^κ, s^λ, t^μ are the maps which include the algebra $\mathbb{P}(\mathbb{F}_2)$ in $\mathbb{P}(\mathbb{F}_4)$ by identifying $(\neg, \wedge)$ with respectively $(\neg_\kappa, \wedge_\kappa)$, $(\neg_\alpha, \wedge_\alpha)$, $(\neg_\omega, \wedge_\omega)$. These three maps play three perfectly symmetric parts, and the union of there images generates $\mathbb{P}(\mathbb{F}_4)$. So the algebra $\mathbb{P}(\mathbb{F}_4)$ looks like a borromean ring made of three logical components, three copies of $\mathbb{P}(\mathbb{F}_2)$; we think of this algebra as the algebra of functions of a borromean logic.

2. *The* borromean analysis *or* logic *in* $\mathbb{P}_4$ *will be the study of expressions for functions as compositions of the three classical logical functions in the images of* r^κ, s^λ, t^μ, *or equivalently the study of compositions of the* 3 *binary Sheffer's functions* $\uparrow_\kappa, \uparrow_\lambda, \uparrow_\mu$, *and subsequently of the* 12 *Sheffer's functions* $\uparrow_\xi$ (*introduced in Proposition* 6.5).
3. *The borromean logic—or the study of elements of* $\mathbb{P}_4$ *as compositions of the* 12 *Sheffer's functions* $\downarrow_\xi$ (*or their dual* $\uparrow_\xi$), *could be reduced to the action of the modality* $(-)^2$ *or of the* 12 *logical speculations* (*see Proposition* 8.7) *on the classical boolean logic* $(\neg_\kappa, \wedge_\kappa)$.
4. *The borromean logic allows an analysis of meanings of paradoxical sentences. We have shown how this possibility is induced from the hexagonal framework on which we can move the logical functions* (*Proposition* 6.4), *and is related to the calculus of logical differential in a bi-hexagonal borromean link of boolean logics* (*Proposition* 6.6).

10. Conclusion: Towards Meanings

In some sense, the Frobenius automorphism $(-)^2$ on $\mathbb{F}_4$ seen here as a modality is an amelioration of the Post cyclic negation (1921), because with it we get a complete system by just adding a boolean structure (not a linear order structure, as in the Post's case). Furthermore, we could also represent $(-)^2$ by combination of several boolean structures, in a very symmetrical way; and of course $(-)^2$ is a part of a field structure. Moreover $(-)^2$ allows the description of the 12 logical speculations or "points of view", and conversely one classical logic $(\neg_\kappa, \wedge_\kappa)$ on $\mathbb{F}_4$ and the logical speculations allow the construction of $(-)^2$, in such a way that $(-)^2$ can be seen as an addition of two points of view.

So, starting from the idea of a logical hexagon, and then of a borromean object, we presented a setting in which the 4-valued logics could be used to treat of paradoxical meanings in classic boolean logic. At the beginning this could be seen as an expansion of an initial idea of Grosjean (1973): here to the idea of splitting paradoxes we added the construction of a *boolean part* ${}^bP$ of any function P on $\mathbb{F}_4$, in a convenient hexagonal and borromean setting. A second source of inspiration for us on the subject of specular logic was the challenge to unified two opposite ideas: the modified negation of Vappereau and the intuitionistic negation of Heyting. In specular logic they appear as respectively a $\sharp$-alteration and a $\flat$-alteration of a classical negation. Here now all that is unified more as being alterations of negations by various *logical speculations*. Then in this setting the idea of Galois and of Boole are linked, and this allows that any function P on $\mathbb{F}_4$ can be presented as a composition of logical operations with respect to several boolean structures (so we get a case of *moving logic*), and then $\mathbb{F}_4$ appears as a boolean logical manifold. But for a given P the construction of a presentation as a composition of logical operations is often tedious, and a question for future investigations in *borromean logic* will be the discovery of *pretty* presentations for any given P, i.e. presentations with a real geometrical signification; then such a pretty presentation will be a real meaning. The best would be that this geometrical view explains a game of alterations by logical speculations, over one classical logic. So in this bi-hexagonal logical geometry of $\mathbb{F}_4$ and in a borromean way, considering $\mathbb{F}_4$ as a logical manifold, a theory of meanings of sentences is possible as a calculus of logical differentials and logical speculations on $\mathbb{F}_4$.

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