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The Mathematical Physics In The Style Of Gabriel Lamé And The Treatise Of Emile Mathieu

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Abstract. The *Treatise of Mathematical Physics* of Emile Mathieu, edited from 1873 to 1890, provided an exposition of the specific French "Mathematical Physics" inherits from Lamé, himself heir of Poisson, Fourier, and Laplace. The works of all these authors had significant differences, but they are pursuing the same goal, described here with its relation to the Theoretical Physics.

1 The "Mathematical Physics" of Laplace, Fourier, Poisson

In the end of the XVIIIth century, with Lagrange and Laplace specially, the undisputed centre of European Mathematical Physics was Paris and this predominance goes on until around 1830 (Greenberg, 77). Then the situation declined, and even some historians as Herivel considered that there was no great French theoretical physicist in the period 1850–1870 (Herivel, 130).

Nevertheless, Grattan–Guinness observed that, if we view the history of physics as not only the history of conceptual innovation and experimentation, but also as an history of engineering applications then French physics is very active after 1830: Navier, Poncelet, Clapeyron, Coriolis (Grattan–Guinness 1990). The question is to decide if inventions in pure mathematics or in engineering are or are not creative physics (Grattan–Guinness 1993). Moreover Grattan–Guinness imagine that when, following the very–mathematical scheme of the mechanico–molecular method of Laplace, as it is the case in French school (see hereinafter), the complexity of the mathematical topics increased, then scientists devoted more and more energy to mathematical issues, at the expense of experimental investigation.

In Physics at the end of the XIX century, in the hands of Poincaré, Theoretical physics is neither a simple application of mathematics, nor, after some mathematical modelling, a final reduction of the understanding of nature to mathematical challenges; its aim is the representation and explanation of observed physical phenomena in nature. In theoretical physics the basic point is to distinguish principles, in a very narrow relation with experimental observations. For Poincaré, in 1904, there are five or six such principles: conservation of energy, energy degradation, equality of action and reaction, relativity, conservation of mass and the least action (Poincaré, 126–127). Mathematical Physics is an analytic framework for the description of physical theories, and also an ideal of understanding. In its broad sense, it is mainly characterized by the use of representations by partial differential equations.

The strict Mathematical Physics is not the same as theoretical physics, and through the Mathematical Physics, the pure analysis and the physics are intertwined, in tension, this is clear in the mind of Poincaré, which worked in both areas (Paty). So, with in mind such a distinction and a tension between theoretical and Mathematical Physics, we could understand the position of Herivel as a valorisation of "creative" theoretical physics, strongly related to experiences (forgetting this tension), and the one of Grattan–Guinness as a reminder of this tension.

Concerning the French theoretical and mathematical Physics, we have to distinguish between two tendencies: one is the mechanico–molecular (Laplace, Ampère, Poisson); the other is the physico–analytic (Fourier). Of course both tendencies use basically of mathematics and of modelling by partial differential equations, in such a way that physical problems will be solve by integrating these equations.

For the physico–analytic tendency, physical laws derived from observation are of primary importance, and are not to be explained in simpler terms, the primary causes will stay unknown. For instance for Fourier the theory of heat have to be based on the law of the action of heat, and the subject of heat by itself is a separate branch of physics, with no necessary connection with dynamics and inter–molecular forces for instance. So the phenomenon of heat is a category which cannot be *sui generis* reduced to any other category.

> Whatever is the extension of mechanical theories, they don't apply to phenomena of heat, which are of a specific nature, out of explanation by movements and equilibrium[1].

And the same for other category of physical subjects (light, electricity, and so on):

> [...] physics by the variety and complication of its phenomena will always evidently be very inferior to astronomy whatever its future progress may be.[2]

Thus the analytic theory of heat starts with the following argument:

> The actions of heat are related to some constant laws, which cannot be discovered without the help of mathematical analysis. The goal of theory exposed here is to demonstrate these laws; it reduces any physical inquire, on the subject of heat, to questions of integral calculus on element given by experience.[3]

For the mechanico–molecular tendency in Lagrange's views, mechanics is at first only reduced to a matter of mathematics, to pure mathematical relations and calculations, "replacing the physical linkages of bodies by equations between the coordinates of their various points", in such a way that physical problems are reduced to a "point of analysis". Then, against this approach, Laplace returned to a dynamical perspective, based on the use of force. Robert Fox showed that in the period 1805–1807 (Fox, 100), Laplace gave his energies almost completely to investigation in molecular physics. As expressed by Poisson in his "Dissertation on elastic bodies read before the Académie on 24th November 1828:

> Beside this admirable conception [of Lagrange] one can now place physical mechanics, whose unique principle is to connect everything by molecular attractions.[4]

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[1] Fourier, ij–iij.

[2] Comte, 429.

[3] "Les effets de la chaleur sont assujétis à des lois constants que l'on ne peut découvrir sans le secours de l'analyse mathématique. La Théorie que nous allons exposer a pour objet de démontrer ces lois ; elle réduit toutes les recherches physiques, sur la propagation de la chaleur, à des questions de calcul intégral dont les elements sont donnés par l'expérience" (Fourier, 1).

[4] Quoted in Herivel, 123.

Siméon–Denis Poisson was a student of Fourier, a very friend of Biot, and Arago wrote that Lagrange "assigned to Poisson a place among the Huyghens, Newton, d'Alembert, Laplace" (Arago, 674), and so its evaluation of the issues is of great importance. As quoted by J. R. Hofmann

> [...] the goal for Laplace, Biot and Poisson was to account for physical phenomena in terms of central forces acting between material particles, particles of light, and the other imponderable fluids of heat, electricity, and magnetism.[5]

So, for this laplacian school, the various physical disciplines, not yet explained in the fundamental framework of forces and particles, have to be considered as "sciences of waiting" or "partial theories" (in the Lamé's words). So, on the subjects of heat, electricity and light, Lamé wrote:

> It will be against evidence to not admit that these three partial theories progress toward a common unique source, a general theory of which they will appear as corollary or particular chapters.[6]

However, between the two attitudes (physico–analytic or mechanico–molecular) Poisson did not choose definitely, and he used both. So he wrote:

> The application of mathematical analysis to questions in physics is based, in each case, on a certain number of laws obtained from observation or, in their absence, on hypotheses that one wishes to verify. The mathematical calculation plays no part in these suppositions. It serves only to develop their consequences to the greatest possible extent, and consequently it offers the best method for comparing theories with experimental results, from all points of view.[7]

Here Poisson is examining the theory of heat, and he adopted a kind of axiomatic point of view (far from the mechanico–molecular approach, and

[5] Hofmann, 295.

[6] Lamé 1840, 2–8.

[7] "L'application de l'analyse mathématique à des questions de physique est fondée, dans chaque cas, sur un certain nombre de lois données par l'observation, ou, à leur défaut, sur des hypothèses que l'on veut vérifier ; le calcul n'ajoute rien à ces suppositions ; il sert seulement à les déveloper jusque dans leurs moindres conséquences, et par conséquent il offre le moyen le plus propre à comparer, sous tous les points de vue, les théories à l'expérience." (Poisson 1815, 435).

near to the Fourier's philosophy), based on three observations on which only the calculations have to be founded (Arnold). In fact, Poisson had the vast project of a complete *Treatise of Mathematical Physics* (not achieved), in which "Pure analysis is not the goal, but the tool; application to phenomena is the essential purpose" (Hermite). In the foreword of the second edition of his *Treatise of Mechanics* of 1833, Poisson wrote:

> Its main goal is to be an introduction to a Treatise of Mathematical Physics, of which my 'new theory of capillary action', published one year ago, is already a part; the other parts will be my various memories on equilibrium and movement of elastic bodies and fluids, or on imponderable fluids, that I project to collect and to complete as well as possible.[8]

Later he included his book *Mathematical theory of heat* of 1835 (with *Treatise of Mathematical Physics* as its primary title page) as a chapter of this project (Poisson 1835). In this book he achieved the historical references by the indication of the work of Lamé on the law of temperatures inside an homogenous ellipsoid with the help of elliptic functions. At this end of the introduction he expressed that this book is the second part of the *Treatise of Mathematical Physics*, in which, with no specific order, he planed to consider the various questions in physics to which he could apply analysis. In the previous page he specified that:

> It will be a matter of deducing, by rigorous analysis, all the consequences of a general hypothesis on the communication of heat, founded on experiment and analogy. These consequences will come from a transformation of the hypothesis, to which calculations add or subtract nothing; and then their perfect conformity with observed phenomenon proof without any doubt the veracity of the theory.[9]

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[8] "Sa destination principale est de servir d'introduction à un Traité de Physique mathématique, dont la Nouvelle théorie de l'Action capillaire, que j'ai publiée il y a un an, est déjà une partie ; les autres parties se composeront des différents Mémoires que j'ai écrits, soit sur l'équilibre et le mouvement des corps élastiques et fluids, soit sur les fluides impondérables, et que je me propose de réunir et de render aussi complets qu'il me sera donné de le faire." (Poisson 1833, *Foreword* (not paginated)).

[9] "Il s'agira de déduire, par un calcul rigoureux, toutes les consequences d'une hypothèse générale sur la communication de la chaleur, fondée sur l'expérience et l'analogie. Ces consequences seront alors une transformation de l'hypothèse meme, à laquelle le calcul n'ôte et n'ajoute rien; et leur parfaite conformité avec les phénomènes observes ne pourra laisser aucun doute sur la vérité de la théorie." (Poisson 1835, 5).

Let us observe that the first sentences of the introductory volume (the *Treatise of Mechanics*) of the project are:

> Matter is whatever which affect our senses anyhow. Bodies are parts of matter limited in any direction, and so they have a shape and a volume. The mass of a body is the quantity of matter in it.[10]

This basic datum of a body, with the consideration of a potential in it, will be the starting point in the Lamé's approach.

2 The "Mathematical Physics" of Gabriel Lamé

For Lamé, the thinking of a physical theory is closely linked with the necessity of mathematics, as he explained in his *Courses of physics of the École Polytechnique*, firstly edited in 1836 (Locqueneux). For him,

> […] the laws of a physical theory have to be the corollaries of a single law; but the discovery of this law cannot be other than the result of the reasoning, and it is here that the mathematical analysis becomes essential.[11]

The process consists to start from an hypothesis, to translate it in an algebraic language and to obtain mathematical formulas. As these formulas can indicate new facts that the physicist can verify, it results incontestable proofs of the reality of the initial hypothesis.

The conception of Mathematical Physics is expressed along the Lessons written by Lamé from 1852 to 1861 in four books: *Lessons on mathematical theory of elasticity* (1852), *Lessons on the inverse functions of transcendents and isotherm surfaces* (1857), *Lessons on curvilinear coordinates and their applications* (1859) and *Lessons on the analytical theory of heat* (1861). In the beginning of the preface of his Lessons on elasticity, he proposed a kind of programme:

> The Mathematical Physics, in itself, is a very modern creation, which exclusively belongs to the Geometers of our century. Today this science contains only three chapters, more or less ranged, which are treated in a full rational way; that means which only lean on unquestionable principles and laws. These chapters are: the theory of static electricity on the surface of conducting bodies; the analytical theory of heat; the mathematical theory of

[10] Poisson 1833, 1.
[11] Lamé 1840, 8.

> elasticity of solid bodies.[12]

He noticed that the last chapter is the most difficult but also the most useful for the industrial practice. He also explained that the Analysis will not take a long time to embrace other parts of the general Physics, like the theory of light or the electro–dynamical phenomena.

According to Lamé, "the true Mathematical Physics is a science as rigorous as the rational Mechanics". So, it is important to distinguish this rational Physics from the applications, which lean on uncertain principles or on empirical formulas. The empirical and partial theories are useful by their applications, they are not yet sciences but they will be soon. So, the purpose of his Lessons is to teach the "true Mathematical Physics" to the students which are the future engineers.

The unity, which is the future of the rational Physics, will be obtained thanks to the Mathematical Physics. In this sense, one of the most important results of Lamé is his theory of curvilinear coordinates, presented in the *Lessons on curvilinear coordinates and their applications* of 1859. In the "preliminary discourse" of these Lessons, he explained the necessity of "a Geometry considered with the point of view of the Mathematical Physics" (Lamé 1859, v), studying a family of curves linked by a common property. Indeed, the purpose of the science of hydrostatics is to determinate the surfaces of the same level of pressure, in celestial mechanics it is the surfaces of the same potential and in the analytical theory of heat it is the surfaces of the same temperature. The theory of light introduces the surfaces of waves and the mathematical theory of elasticity introduces three families of conjugate and orthogonal surfaces. From this, comes the idea of the curvilinear coordinates, which is essential in all the fields of the Mathematical Physics:

> In all these fields, the question is always to integrate or to determine functions, which has to verify one or several partial differential equations of the second order, expressing the physical laws which govern the functions. Furthermore, these functions, or their general integrals, have to satisfy to other partial differential equations of the first order, at each point

[12] "La Physique mathématique, proprement dite, est une création toute moderne, qui appartient exclusivement aux Géomètres de notre siècle. Aujourd'hui cette science ne comprend en réalité que trois chapitres, diversement étendus, qui soient traités rationnellement ; c'est-à-dire qui ne s'appuient que sur des principes ou des lois incontestables. Ces chapitres sont : la théorie de l'électricité statique à la surface des conducteurs ; la théorie analytique de la chaleur ; enfin la théorie mathématique de l'élasticité des corps solides." (Lamé 1852, v).

> of the boundary of the treated body. But this problem of double integration would be completely inaccessible without the help of a convenient system of coordinates, such that the surface is expressed by the fact that one of these coordinates is a constant.[13]

In some sense, it is near of the conception of Leibniz and his friends Bernoulli to solve physical problems, like the catenary, the isochron curve or the bratystochron curve. About the catenary, Leibniz wrote to John Bernoulli in 1695:

> It is not always easy to reduce the [physical] problems to an inverse problem of tangents. For instance, for the research of the catenary, if we did not know a property of its tangents by mechanical theorems in relation of its center of gravity, it would be difficult to find the construction.[14]

So, for Lamé as for Poisson, a physical phenomenon is always the data of a given body equipped with a function. From this point of view, the mathematical theory of curvilinear coordinates plays a major role. It is also one of the most important Lamé's contributions to the mathematics of surfaces (Chasles 146–149). The mathematics of Lamé constituted an interesting dialectics between algebra and geometry (Barbin).

This introduction of curvilinear coordinates allows to unify the different physical parts. The function is the potential in the theory of attraction, and the function is the temperature in the theory of heat, but if the temperature is independent of time then the general partial differential equation of the second order is the same for these two theories. As well, "the theory of potential sways between two analogies: the first one is the hydrostatics and the second is the theory of heat" (Lamé 1859, ix). Finally, the isostatic systems correspond to the orthogonality of three families of surfaces governed by a partial differential equation of fourth order. Lamé wrote the wanted unity with enthusiasm:

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[13] "Dans toutes ces branches, il s'agit toujours d'intégrer, ou de déterminer, une ou plusieurs functions qui doivent verifier une ou plusieurs équations aux différences partielles du second ordre, exprimant les lois physiques qui régissent les fonctions dont il s'agit. Et en outre, ces fonctions, ou leurs integrales générales, doivent verifier d'autres équations aux différences partielles du premier ordre, pour tous les points appartenant à la surface qui limite le corps que l'on veut traiter. Or ce problème de double integration serait complètement inabordable, si l'on ne parvenait pas à rapporter les points du corps à un système de coordonnées tel que la surface, ou les diverses parties qui la composent, soient exprimées par une de ses coordonnées égalée à une constane." (Lamé 1859, viii).

[14] Leibniz 1989, 190.

> New connection which foresees the future advent of an unique rational science, containing by its formulas, the three fields of applied mathematics, that I defined, and, moreover, the theory of sound waves and light waves, which are not something else than the elasticity in the dynamical state.[15]

In his *Dissertation on the isotherm surfaces in solids* of 1837, Lamé defined a family of "isotherm surfaces" (Lamé 1837) as a family of surfaces

$$\lambda(x, y, z) = \lambda_0$$

such that it exists a function V(x, y, z) which the values depend only of λ(x, y, z), so that

$$V(x, y, z) = V_0$$

is the initial family and V(x, y, z) has a laplacian equals to zero ($\Delta V = 0$). Therefore, V can be considered as a temperature or a thermic parameter. Lamé showed that V can be expressed as a function of λ:

$$V(\lambda) = A \int_a^{\lambda} d\lambda/\varphi(\lambda) + B$$

with

$$\varphi(\lambda) = \exp\left(\int g(\lambda)d\lambda\right)$$

and

$$g(\lambda) = (\Delta\lambda)/(\lambda'_x + \lambda'_y + \lambda'_z)^2.$$

He emphasized the interest of the curvilinear coordinates in space and the notion of a triple orthogonal system. Three families of surfaces

$$f_1(x, y, z) = h_1,\quad f_2(x, y, z) = h_2, f_3(x, y, z) = h_3,$$

constitute a "triple orthogonal system" if by each point of space, it passes one surface of each family and if at this point these surfaces are orthogonal. Following the Dupin's theorem (Dupin), these surfaces cut each other along the curvature lines. The fundamental theorem is given in

[15] "Nouveau rapprochement qui fait entrevoir l'événement futur d'une science rationnelle unique, embrassant, par les mêmes formules, les trois branches des mathématiques appliqués, que je viens de définir, et en outre, la théorie des ondes sonores et celle des ondes lumineuses, qui ne sont autres que la théorie générale de l'élasticité dans l'état dynamique." (Lamé 1859, x).

the *Dissertation on orthogonal and isotherm surfaces* of 1843: the only isotherm and triple orthogonal systems are the systems of confocal quadrics (Lamé 1843).

In his *Lessons* of 1859, Lamé especially studied the case of ellipsoidal coordinates (Lamé 1859). He introduced three families of surfaces: ellipsoids, hyperboloids with one sheet and hyperboloids with two sheets, which have the respective equations:

$$x^2/\rho^2 + y^2/(\rho^2 - b^2) + z^2/(\rho^2 - c^2) = 1$$
$$x^2/\mu^2 + y^2/(\mu^2 - b^2) - z^2/(c^2 - \mu^2) = 1$$
$$x^2/\nu^2 - y^2/(b^2 - \nu^2) - z^2/(c^2 - \nu^2) = 1$$

with $c > b > 0$ and $\rho > c > \mu > b > \nu > 0$. A point (x, y, z) is located by the three geometrical parameters ρ, μ, ν, which are called “elliptical” or “ellipsoidal coordinates”. He showed that these three families form a triple orthogonal system, then that the three functions written with the help of the elliptical functions:

$$\xi = \int_c^\rho d\rho/\sqrt{(\rho^2 - b^2)}\,\sqrt{(\rho^2 - c^2)},$$
$$\eta = \int_b^\mu d\mu/\sqrt{(\mu^2 - b^2)}\,\sqrt{(c^2 - \mu^2)},$$
$$\zeta = \int_0^\nu d\nu/\sqrt{(b^2 - \nu^2)}\,\sqrt{(c^2 - \nu^2)},$$

are three thermic parameters and this proves that the system is isotherm (Guitart).

3 The unity of Physics: the aether and its equation

The purpose of Lamé in his *Dissertation on the laws of equilibrium of the aetheral fluid* of 1834 was to find the differential equations of the light (Lamé 1834). He notices that the works of Fresnel made sure the existence of an universal fluid where the light waves are propagated. But some difficulty stays because the laws which govern the “aetheral fluid” are not known. He explained that, because of the nature of aether, it is not possible to know these laws by experiences. But we can hope to find them by applications of mathematics to the complex phenomena whose these laws are the causes. He proposed to start with two general facts, to take one of them as a “fundamental principle” and to deduce the other one by the calculus.

Accordingly with experiences on interferences of Fresnel on light, he took as a fundamental principle, that “the light is caused by the vibrations of aether without change of density" (Lamé 1833, 194). As general phenomenon, he chose the fact that there exist translucent bodies, weighted mediums in which light waves are propagated. He has to calculate the law of distribution of molecules in the neighborhood of a vibrating molecule. Finally, he obtained the differential equations, which represent the vibrations of the light in the aether and the general equation of the equilibrium of the aether, on the surface or inside the translucent bodies. This equation is (Lamé 1834, 213):

$$d^2 \log \rho / dx^2 + d^2 \log \rho / dx^2 + d^2 \log \rho / dx^2 = 0.$$

He integrated the equation with a special case of particular coordinates. For Lamé, the aether is the "truly universal principle of nature", as he wrote in his *Note on the working to follow to discover the only true universal principle of Nature* (Lamé 1863). He explains that

> […] constant works led him to a kind of a new definition of the Mathematical Physics, to the prediction of the true aim towards this general science converges.[16]

He began the paper by a historical outline with six successive fields worked: capillarity, electricity, magnetic, propagation of heat, of light, and elasticity of bodies. From this, he announced three predictions. Firstly, the principles of the three first fields will be reached when we will know those of the last three fields. Secondly, the theories of elasticity and light have to be melt. Thirdly, it will stay only two theories and Lamé concluded that the truly principle of physical nature will come from their fusion.

But the propagation of the light in the vacuum and in planetary spaces, and the phenomenon of interferences unquestionably show the existence of an aetheral fluid. For Lamé, this kind of matter is more universal and active than the weighted matter. So, […] the future sciences will recognize in the aether the true king of the physical nature"[17] and " the cristallography, where Fresnel created the theory of light, will be always the laboratory that we have to choose to further the general science.[18]

Lamé explained that he recently proved new results on propagation of light in translucent bodies by using the theory of elasticity "created by Clapeyron". For him, this "theoretical extension" gave the only rigorous proof of the existence of free aether in translucent bodies.

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[16] Lamé 1863, 983.
[17] Lamé 1863, 986.
[18] Lamé 1863, 987.

The "Note" of 1863 shows that his works on Mathematical Physics and the existence of aether are two faces of the same Lamé's thought on the unity of physics. This claimed unity, associated to the name of his friend Clapeyron, seems to be as an echo of the saint–simonian philosophy, which impregnated the two scientists in their youth (Régnier). Lamé wrote that he thought that he was the only geometer who worked on this kind of questions but many communications of the Académie des sciences showed to him that there exist possible colleagues in Switzerland, in Germany, in Austria. So, he concluded the "Note" by a kind of last will and testament:

> The true tendency of the physical-mathematical work of our century being admitted, it was important to well define the present state and to prepare the future. Most of the workers of the already made work are disappeared and I am the oldest of those which stay. Before to live the place, I thought that I have a duty to fulfill, the duty to collect, to purify, to simplify the obtained results, with the aim to facilitate to our successors the completion ot the total work. It was the purpose of the four Courses that I published successively. The following Course had to summary the others, in a more concise but also in a more complete form; but I am conscious of the strength and of the time, which will miss to finish this last Course, which the present note had to serve as an introduction.[19]

4 Emile Mathieu and his project of Treatise

Emile Mathieu was born in 1835 in Metz. He entered in École Polytechnique in 1854, and five years later, he defended before the university La Sorbonne (Paris) a thesis in Algebra on the number of values of a function and on transitive functions (Floquet 2). Along his life he worked on many fields of mathematics: algebra, theory of numbers, integral calculus, elliptic functions, celestial mechanics and analytical mechanics. In 1866, Gabriel Lamé was too ill to give the Course on

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[19] "La véritable tendance de l'oeuvre physico-mathématique de notre siècle étant reconnue, il importait de bien définir son état présent et de préparer son avenir. La plupart des ouvriers du travail déjà exécuté n'existent plus, et je suis le doyen de ceux qui restent. Avant de quitter cette place, j'ai pensé que j'avais un devoir à remplir celui de recueillir, de purifier, de simplifier les résultats obtenus, afin de faciliter à nos successeurs l'achèvement de l'oeuvre totale. Tel a été le but des quatre Cours que j'ai successivement publiés. Le suivant devait les résumer tous, sous la forme la plus concise et en même temps la plus complète; mais je sens que les forces et le temps me feront défaut pour terminer ce dernier Cours, auquel la Note actuelle devait servir d'introduction." (Lamé 1863, 989).

Mathematical Physics of La Sorbonne, and he proposed him to be his substitute. This proposal was not accepted by the Minister Victor Duruy, but, in 1867, Mathieu became "free teacher" at La Sorbonne. His Course "Mathematics" was announced in the *Revue des cours scientifiques* as a course on *Methods of integration of Mathematical Physics*[20]. It will become the first volume of his *Treatise on Mathematical Physics*. Then he obtained a chair of Professor in the Faculty of Besançon and finally in the Faculty of Nancy in 1873 where he staid until his death in 1890.

In 1863, Mathieu gave a first note on Mathematical Physics, "on the law of liquids through tubes of very small diameter" in the *Comptes rendus de l'Académie des Sciences* (Duhem, 158). During twenty years, he published almost twenty papers on Mathematical Physics, where he shown himself to be the continuator of Laplace, Fresnel, Poisson and Lamé, he also constructed his own tools and programme and he solved difficult problems.

The first paper appeared in 1866 in the *Journal de mathématiques pures et appliquées*, on the dispersion of light. After having mentioned the works of Poisson and Cauchy, Mathieu came on the way shown by Fresnel and Lamé. In the *Dissertation on the double refraction* of 1821, Fresnel in 1821 obtained the equation of a surface of a wave of light by composition of vibrations in an aetheral fluid. So, in his book of 1852, Lamé explained that the theory of the elasticity has to be applied to the light (Lamé 1852). Mathieu wrote: "this work of M. Lamé is the basis of our dissertation" (Mathieu 1866, 51). Indeed, he began to give a new form for the equations of elasticity, later he shown the advantage of this form and then he gave the complete answer for the dispersion of the light in uniaxial crystals.

Mathieu came back to the theory of the elasticity in a paper of 1868 of the same *Journal*, with the difficult subject of the vibrations of an elliptic membrane. The cases of a rectangular and of an equilateral triangular membranes were already studied by Lamé, and the case of a circular membrane was also by Bourget. The purpose of Mathieu is to determinate "by the analysis" all the circumstances of the oscillatory motion of a membrane subjected to an equal tension in all directions. He began to describe the results of the experiences obtained by Félix Savart in 1840 for an elliptic membrane that is two systems of nodal lines, which are ellipses or hyperbolas having the same focus than the ellipse of the membrane. As he wrote about these experiences: "the Mathematical Physics has to give an account of the facts of experience" (Mathieu 1869b, 257). To solve "by

[20] The Course was every Monday and Wednesday afternoon and began in November*, Revue des cours scientifiques de France et de l'étranger*, 1867– 1868, 5, 832.

the analysis" the case of an elliptic membrane, he introduced the differential equation nowadays named "equation of Mathieu" (Mathieu 1868, 146):

$$d^2P/d\alpha^2 + (N - 4\,\lambda^2\,c^2\cos^2\alpha)\,P = 0.$$

He arrived to this equation by separating the variables in the equation of vibrating membrane, written with elliptic coordinates.

After a paper on the motion of the temperature, where he used the curvilinear coordinates of Lamé to solve the difficult problem of solids limited by circular cylinders and lemniscatical cylinders, Mathieu came back to the theory of elasticity, but to give a parallel study on two theories : the theory of potential and the theory of elasticity. It is the start of his important paper of 1869: "Dissertation on the partial differential equation of the fourth order $\Delta\Delta u = 0$ on the equilibrium of elasticity in a solid". He explained that

$$\Delta v = 0$$

is the equation of the second order of the potential and that his purpose is to study the properties of the equation of the fourth order

$$\Delta\Delta u = 0.$$

He wrote:

> This equation can be found in Mathematical Physics […] when an homogeneous solid, which has the same elasticity in all the directions, is submitted on his surface to tensions which keep its in an equilibrium of elasticity. […] So we understand the interest to study the function, which satisfy this equation and we will see that this study will permit to integrate it.[21]

[21] "Cette équation se rencontre en physique mathématique ; […] quand un corps solide, homogène, et dont l'élasticité est la même dans tous les sens, est soumis à la surface à des pressions qui le maintiennent en équilibre d'élasticité. […] On comprend donc l'intérêt qu'il y a à s'occuper de la fonction qui satisfait à l'équation, et nous ferons voir d'ailleurs que cette étude permettra d'intégrer cette equation." (Mathieu 1869c, 378).

In the first part of the paper, he gave the Poisson's formula for the equation of potential

$$\Delta v = 0$$

and stated that there always exists one and only one finite and continuous function *v* with continuous derivatives inside of the surface s, which satisfies $\Delta v = 0$, and which has a given value on the surface s. He pointed that this theorem on potential is not easy to prove, but it is almost obvious for the equilibrium of temperature, which is submitted to the same equation $\Delta v = 0$. He mentioned that Green also proved the theorem by using the theory of electricity. He concluded that "many theorems on the potential became intuitive by substituting to them those of the equilibrium of temperature" (Mathieu 1869c, 384).

In the second part of the paper he came to the equation $\Delta\Delta u = 0$ and gave the concepts of first potential and second potential, already introduced by Lamé in his Theory of elasticity (Lamé 1852, 70–71). He considered the potential given by the triple integral

$$v = \int\int\int \phi\,(a, b, c) / r \; da\; db\; dc$$

where r is the distance of the point (x, y, z) to the variable point (a, b, c). We have

$$\Delta v = 0 \text{ or } \Delta v = -\,4\,\pi\,\phi(x, y, z)$$

depending on whether that the point (x, y, z) is inside or outside of the volume. Then he considered the function:

$$w = \int\int\int\; r\,\phi(a, b, c)\;\; da\; db\; dc$$

and as Lamé he obtained:

$$\Delta w = 2v.$$

So, we have:

$$\Delta\Delta w = 0 \text{ or } \Delta\Delta w = -\,8\,\pi\,\phi(x, y, z)$$

depending on whether the point (x, y, z) is inside or outside of the volume. He called *w* the "second potential" and *v* the "first potential". He proved that it is always possible to find a function *u* such that

$$\Delta\Delta u = 0$$

inside a surface σ, which is continuous, as well as its three derivatives up to the third order, and such that its value and the value of its Δ are given on the surface. The paper goes on with the boundary conditions which perfectly determine the solution of $\Delta\Delta u = 0$.

The paper *Dissertation on the integration of the partial differential equations of the Mathematical Physics* of 1872 can be considered as the first one where Mathieu gave a general conception on Mathematical Physics. He wrote:

> The principal partial differential equations that we meet in the Mathematical Physics are
> $$\Delta u = 0,\ \Delta\Delta u = 0,\ \Delta u = -\,a^2 u,\ du/dt = a^2 \Delta u,\ d^2u/dt^2 = a^2 \Delta u,$$
> where t is the time. The function u, which represents a temperature, a potential or a molecular motion, satisfies one of these equations inside a solid limited by a surface σ or inside a plane limited by a line σ. Moreover, u and its derivatives of the first order are to be continuous in this space.[22]

The five equations are well known as corresponding to various physical phenomena, but the paper does not recall them systematically. Mathieu began to come back to the parallelism between the two cases

$$\Delta u = 0,\ \Delta\Delta u = 0,$$

with two theorems. The first theorem stated that every function u of x, y, z which satisfies the equation $\Delta u = 0$ inside a volume limited by a surface σ, and which is continuous, as well as its derivatives of the first order, can be considered as the potential of an infinitely thin layer distributed on the surface s. The second theorem stated that every function u of x, y, z which satisfies the equation $\Delta\Delta u = 0$ inside a surface σ, and which is continuous, as well as its derivatives of the three first order, is the sum of the first potential of an infinitely thin layer distributed on s and of the second

[22] "Les principales équations aux différences partielles que l'on rencontre dans la Physique mathématique sont les suivantes
$$\Delta u = 0,\ \Delta\Delta u = 0,\ \Delta u = -\,a^2 u,\ du/dt = a^2 \Delta u,\ d^2u/dt^2 = a^2 \Delta u,$$
dans lesquelles *t* désigne le temps. La fonction *u*, qui représente une température, un potentiel ou un déplacement moléculaire, satisfait à une de ces équations dans l'intérieur d'une corps déterminé par une surface σ ou dans l'intérieur d'une surface plane limitée par une ligne *s*. De plus, *u* et ses dérivées du premier ordre doivent varier d'une manière continue dans cet espace." (Mathieu 1872a, 249).

potential of a similar layer distributed on the same surface σ. Mathieu linked the first theorem to the work of Green on electricity, and the second theorem to his own work on elasticity.

The purpose of this paper is to solve all these partial differential equations of the Mathematical Physics in solids of any forms. To solve them completely, it is necessary to give one or two boundary conditions. This conception is completely in accordance with the spirit of the Course given in La Sorbonne in 1867–1868. Indeed, in the same year 1872 and in the same *Journal de mathématiques pures et appliquées*, Mathieu wrote a short paper titled *On the publication of a course on Mathematical Physics given in Paris in 1867 and 1868* where he explained:

> The solutions of the partial differential equations in Physics present a particular character, which distinguishes them of the solutions that we meet in the other fields of Mathematics. Generally, the intended functions not only satisfy to these equations inside a surface, but, moreover, on this surface they satisfy to certain equations that we called boundary conditions [...]. After that the analytical form of a problem of Mathematical Physics was well specified, it is obvious that we could substitute to this problem a pure analytical question; but we have to note that in this way we would take off this problem something of its interest and its clarity almost always.[23]

Mathieu gave historical examples to show that the series given by the pure mathematicians are not sufficient because for the Mathematical Physics needs to know when these series are convergent. For instance, for the vibrating–string problem, he compared the solution of d'Alembert without fixing the extremities of the string and the best solution of Daniel Bernoulli. He paid tribute to the "remarkable books" published by Lamé, but he pronounced two criticisms: Lamé dismissed the historical part in his books and he did not clarify the results of his predecessors.

The Course is edited in 1873 under the title *Course of Mathematical Physics*. The first sentences of the Preface are:

[23] "Les intégrations des équations aux différences partielles de la Physique présentent un caractère particulier qui les distingue des integrations que l'on rencontre dans les autres branches des Mathématiques. En general, les fonctions que l'on y cherche satisfont non seulement à ces equations dans l'intérieur d'une surface, mais elles satisfont, de plus, sur cette surface, à de certaines equations que l'on appelle les conditions aux limites. […] Après avoir bien precise la forme analytique d'un problème de Physique mathématique, il est evident que l'on pourrait à ce problème substituer une question d'analyse pure ; mais il convient de remarquer que l'on ôterait par là presque toujours de son intérêt et de sa claret." (Mathieu 1872b, 418).

> The Book that we publish could be used as a first volume of a Treatise of Mathematical Physics, which would involve all that you know as the most rigorous in this field of Mathematics. So we would give to the present volume the title: Methods of integration in Mathematical Physics. The treatises related to this subject are: the Analytical theory of heat of Fourier, The mathematical theory of heat of Poisson and the books of Lamé.[24]

We can conclude of this that, in 1873, Mathieu has the project of a complete treatise in mind and that this first volume is related to the most simple equations of the Mathematical Physics. He explained that he will mention the works of all his predecessors:

> We came back on all their works, and we tried to present the state of the Science today in this field of the Analysis. We took care to treat the successive questions with the greater uniformity as possible, to emphasize the methods and to avoid any calculations, which has no mathematical interest. Indeed, as the field of Science grows, it is necessary to state the principles with more clarity and concision and to delete clever calculations and to substitute to them some transformations of whose we have to give an account.[25]

So, the project of the Mathematical Physics is to make uniform the different fields of Physics but also the different works on Physics elaborated at this period. Indeed, Mathieu gave the example of the successors of Fresnel, Mac–Cullagh, Neumann, Lamé, who worked without knowing them and without reading the others.

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[24] "L'ouvrage que nous publions pourrait servir de premier volume à un Traité de Physique mathématique qui renfermerait tout ce que l'on sait de plus rigoureux dans cette branche des Mathématiques. Alors on donnerait au volume actuel ce titre : *Méthodes d'intégrations en Physique mathématique*. Les Traités qui se rapportent à ce sujet sont : la *Théorie analytique de la chaleur*, par Fourier ; la *Théorie mathématique de la chaleur*, par Poisson, et les ouvrages de Lamé." (Mathieu 1873, v).

[25] "Nous sommes revenus sur tout ces travaux, et nous avons cherché à exposer l'état actuel de la Science sur cette branche d'Analyse. Nous avons eu soin de traiter les questions qui se présentent successivement avec le plus d'uniformité possible, de mettre en relief les methodes et d'éviter tout calcul qui soit sans intérêt mathématique. En effet, à mesure que le domaine de la Science s'agrandit, il faut en exposer les principes avec plus de clarté et de concision et supprimer les calculs habiles pour leur substituer des transformations dont on doit rendre compte." (*Ivi*, vii).

The Book is composed of nine chapters. The chapter 1 obeyed an historical order with “the use of trigonometric series” for the vibrating–string and for the heat in a solid. The chapter 2 gave the Lamé’s works “on isotherm surfaces and on curvilinear coordinates” and the next one chapter contained new results on “equilibrium of temperatures in indefinite cylinders”. Then the chapter 4 “on linear differential equations of second order” leaned on the Sturm’s ideas but with simpler theorems. The next chapters came back on these treated subjects: “motion of a vibrating membrane and temperatures of cylinders” in chapter 5; “distribution of a temperature in a sphere” in chapter 6 with Laplace and Poisson’s theorems and “distribution of the heat in an indefinite medium and temperatures of the globe” in chapter 7with simpler results than those of Poisson; “on equilibrium of the temperature of the ellipsoid” in chapter 8. The last chapter 9 “on the cooling of the planetary ellipsoid” contained news results of Mathieu. Many of these chapters contain new proofs or new results provided by Mathieu.

5 The Mathieu’s *Treatise of Mathematical Physics*

In 1883, Mathieu published a book on the theory of capillarity. He presented it as the second volume of a series of volumes constituting a *Treatise of Mathematical Physics*, with a first volume which is the *Course of Mathematical Physics* of 1873. From Floquet, we know that ten volumes had been planned, but at the death of Mathieu in 1890, three volumes are not yet written: on the theories of light (some handwritten notes exist), on the motion of a gas, on acoustics, etc (Floquet, 23-24). None book on thermodynamics was planned. So the existing books are: I. *Course of Mathematical Physics* (1873); II. *The theory of capillarity* (1883); III–IV. *Theory of potential and its applications to electrostatics and to magnetism* (1885–1886); V. *Theory of electrodynamics* (1888); VI–VII. *Theory of elasticity in solid bodies* (1890).

We have to notice that Mathieu published another book in 1878, his very interesting *Analytical dynamics* (Mathieu 1878). But this book was not included as a volume in the series of 1883. Perhaps because it is a part of Celestial mechanics and Astronomy, which are not exactly considered as included in Physics since Lamé. In this book, Mathieu intended to write with a maximum of mathematical analysis and a minimum of principles from geometrical or mechanical reasoning. For the opposite standpoint, he recommended the treatise of Mechanics edited by Résal. He recalled that a nice updating of the Lagrange’s *Analytical mechanics* is provided in the "excellent Notes" of Joseph Bertrand, but he claimed that now all the old results have to be completely taken again with the discoveries of Poisson, Hamilton, Jacobi. It is the purpose of his book. According to Mathieu, his book did not include static and hydrodynamics, and it is a way to emphasize the uniformity of the analytic treatment.

The section 1 on general theorems of dynamics began with the Bernoulli's principle, because "all the science of equilibrium is based on the principle of virtual speeds", as Mathieu wrote. Then he continued with the d'Alembert's principle, and provided a new simple proof of hamiltonian equations. The main discoveries of Lagrange, Poisson, Hamilton and Jacobi are introduced in the next section. The book included the theories of movement of material points, of rotations of a solid body and of relative movements. A theory of perturbations is also given, which contained the general formula for the expression of perturbations obtained by Mathieu himself in 1874 in the *Journal of Liouville*. It finished with a study on projectiles in air, in which he integrated the trajectories "without any special hypothesis on the law of the resistance of the air, which so is let to the choice of the calculator" (Mathieu 1874, 267).

The volume on capillarity began with an historical introduction on the subject, where Mathieu referred to Borelli (1670), with quoting Poggendorff in his *Histoire de la Physique*, translated in French in 1883 (Poggendorf, 249). But for him, the real mathematical theory began with Young and with Laplace in 1805–1806, and then with Poisson and Gauss. In the five chapters, these theories are presented, criticized, compared with the experiences, and often completed and generalized.

Mathieu started with an application of the principle of virtual speeds (as it is expressed in his *Analytical dynamics*) to the description of the Laplace's conception of the force of capillarity. Laplace agreed with the Hawksbee's conception: in a vertical capillary tube, the water rises with a meniscus which has a higher level than the water outside, and this capillarity can be explained by a special attractive force between two molecules m and M. This force is different from the gravity one, but it also acts on the line mM as a function F(r), where r is the distance between m and M, depending on the nature of the matter of m and the matter of M. It is noteworthy that in fact F(r) is not exactly known, just it is assumed that its value is significant and not equivalent to zero for only very small values of r (less than the radius of an activity sphere). More precisely, for Laplace, the hypothesis on F(r) is that

$$\Phi(r) = \int_r^{\infty} F(r)\, dr$$

is almost null if r exceeds the radius of activity.

Mathieu emphasized that these capillarity forces derived from a capillarity potential. He specified that he "extends the word 'potential' to the case of an attraction which is not as the inverse of the square of the distance" (Mathieu 1883, 9), this potential V being of the form

$$V = \iint_{\Sigma} \Phi(r)\, \rho(M)\, d\sigma.$$

So, by integration, Mathieu obtained the corresponding partial differential equation for the surface of a liquid partially free and partially in contact with a solid:

$$Z - h = M\,(1/R_1 + 1/R_2),$$

where R_1 and R_2 are the principal curvatures, expressed by partial differentials of z. Then, he obtained the value of the angle at the junction line, and then the value of the superficial tension. These calculations by Mathieu are always available, even if the density of the liquid is variable.

It results that all the capillarity phenomena could be explained and that these problems are physical problems in the Lamé's style: a partial differential equation and some bounding conditions. All the examples observed by Laplace are again examined here and unified and extended as possible. There are the problems on the rise and depression of a liquid close to the inner wall, on the superposed liquids, on the suspension of a liquid in the air by a capillary tube, on the rise of a liquid by the means of an horizontal disc and on the shapes of drops of liquid put on an horizontal plane or suspended. Mathieu gave a new proof of the Bertrand's result on the volume of a drop. Perhaps the more important new result of Mathieu is an improvement of the Archimedes' principle, which is that any floating object is buoyed up by a force equal to the weight of the fluid displaced by the object, with taking in account the pressure by capillary forces or superficial tension. Mathieu solved this problem for an object of any shape. Before him, this problem was solved by Poisson only in a very special case, as Mathieu wrote:

> In his book [on capillarity], Poisson took the theory with a very difficult point of view, studying the modifications of pressure due to the capillary action on a body embedded in a liquid. In despite his great skill, he solved the problem only in the case of a body of revolution with a vertical axis.[26]

[26] "Dans son livre [sur la capillarité], Poisson a pris cette théorie par un côté très difficilement accessible. Il étudie en effet dans cet endroit les modifications de

The book is not at all an abstract mathematical book, but, starting with physical principles borrowed from its predecessors, Mathieu constructed a very deep calculus in order to model problems with potentials by partial differential equations. Several difficult mathematical challenges are surmounted, but also physical results are obtained, with real qualitative meanings, and are submitted for checking by experiments. Clearly here, Mathieu is a true physicist and proud of to be that. For instance, Mathieu pointed out that he was the first in 1863 to prove that when a liquid pours in a capillarity tube, there is a very thin layer of motionless liquid near the side surface (Mathieu 1883, 50).

Let us remark that the next year Henri Résal, a student of Poncelet and Lamé, published also a book titled Mathematical Physics (Résal 1884), where he claimed that the Mathematical Physics originates from the project of Poisson, and that “its starting point have to be considered as the theory of capillarity as formulated by Laplace”. In 1888, Résal published a Treatise of mathematical Physics in two volumes (Résal et al. 1888–1889).

In the two following volumes (1885–1886) on the *Theory of potential and its first applications*, Mathieu preferred to separate the exposition of the principles from their applications, to obtain a more rigorous account, as he wrote:

> A lot of problems theorems on potential are interesting for the physical properties that they exhibit; but the separation of these theorems from their applications offers the advantage to exhibit them with a complete rigor. These mathematical results, which are born in Mathematical Physics, are now carried in pure mathematics.[27]

He began the chapter 1 with the potential of gravitation in 1/r, the calculus of the corresponding laplacian and the Poisson’s formula. He gave the Green’s formula for the integration of

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la pression, sur un corps plongé en partie dans un liquide, par l’action capillaire. Mais, bien qu’il déploie dans cette recherché la plus grande habileté, il ne parvient à résoudre la question, que pour un corps de revolution dont l’axe est vertical.” (Mathieu 1883, 4).

[27] “Une grande partie des théorèmes relatifs au potentiel prennent leur principal intérêt dans les propriétés physiquesq u’ils démontrent ; mais la séparation des premiers théorèmes de leurs applications a cet avantage de permettre de les exposer plus facilement avec une complète rigueur. […] Les resultants mathématiques, qui sont exposés ici, ont pris leur origine dans la Physique mathématique, masi beaucoup peuvent être transportés, ou l’ont déjà été, dans des recherches de Mathématiques pures.” (Mathieu 1885–1886, *Preface*).

$$U\,\Delta V - V\,\Delta U,$$

and he pointed out that this formula was employed by Fourier and Poisson for the cooling of a body, a long time before Green wrote his paper on electricity. Then he presented the Dirichlet's principle on the minimization of energy. The next chapter is devoted to the potential of a thin layer of matter on a surface and to the problem of the determination of the functions, which could be expressed as such a potential. He introduced the Green's function as the key process for such "inverse" potential problems. Nowadays, these two chapters could be yet considered as an excellent clear introduction to the classical potential theory.

In the third chapter, Mathieu reproduced some of its own results published in the *Journal de Liouville*, on logarithmical potential (in log r), on calorific potential (in cos(ar) / r), on second potential (in r). As Mathieu wrote, the properties of these potentials are analogous to the properties of the potential of gravitation, and they are present in the Mathematical Physics. Then he explained how the theory of potential is equivalent to the theory of heat in the case of stable situations of equilibrium of temperature. He also gave the Duhamel's equation of the heat in crystallized body and the corresponding potential (Duhamel). The final chapter is devoted to the attraction of ellipsoids, the Legendre's theorem, and the calculus of the potential of an ellipse, according to Betti.

In the volume on electrostatics, at first Mathieu referred to Poisson's works. We also know that Poisson was one of the primary references for Green. Basically, he began as in the case of gravitation, with a potential in 1/r, but with a density of charge, which could be positive or negative. Furthermore, the charges could move in conductors and remain in equilibrium at the surfaces. Mathieu studied the distribution of electricity on a conical conductor and the mutual influences of two spherical charges. For the theory of dielectrical, Mathieu adopted the starting point of Maxwell, but on the contrary of him, he claimed that the deformation of the dielectric could not be assimilated with the deformation of an isotropic solid (Mathieu 1885–1886, 110). For the theory of magnetism, the starting point of Mathieu is the Poisson's theory with two magnetic fluids. He also used the Coulomb's conceptions: the difference between magnetism and electricity is that, in the case of electricity, there are not free magnetic charges but only magnets (dipoles) and double layer distributions (*Ivi*, 149). With respect to that, Mathieu modified the theory of Poisson, with different physical ideas but with the same equations for the magnetic induction (*Ivi*, 155). For instance, he did not admit the division of a magnetic body in separated magnetic particles.

The next volume, edited in 1888, was on the *Theory of electrodynamics*. In the Mathieu's view, the central thinking is a continuation of the volume on potential, namely the thinking that if a permanent current goes through a conductor, then its edging surfaces are covered with a double layer of electricity. In the chapter 1, he gave the general principle on the movement of electricity inside a conductor, and, in the other chapters he studied the particular results obtained by many physicists. In the chapter 2, he exposed the Kirchhoff and the Ampère's results on permanent linear currents, and, in the next chapter, the induction is introduced according to Weber, Helmholtz, Neumann and Maxwell. In the chapter 4, Mathieu exposed his theory of the double layer, in agreement with Kirchhoff's views. The other chapters are devoted to the permanent currents in plates, the electric units, the movements of electricity in conductors of arbitrary shapes, and finally the telegraphic wires.

The final two volumes dealt with the subject of elasticity in solid bodies. On the one hand, the initial purpose was intended to the art of engineers, with the approximated calculus of strength of materials. But on the other hand, the elasticity is considered as a fundamental question in physics, for instance the elasticity of aether explains the theory of light, the actions of the electrical particles and of the celestial bodies (Mathieu 1890, 1–2).

The chapter 1 began with the Lamé's determination of the ellipsoid of elasticity at each point of a body (nowadays it is identified with the elasticity tensor), and the differential equations of elasticity are presented in several ways. The chapter 2 ended with a proof that the system of elasticity forces inside a body, even isotropic, cannot be considered as a system of attractions and repulsions among molecules following a function of distances. The Saint–Venant's theory of torsion and flexion of cylinders is the purpose of the chapter 3. In the chapter 4, following Lamé, the equations of elasticity in curvilinear coordinates are introduced, but with simpler calculations. Precisely, Mathieu worked with a family of surfaces and its orthogonal trajectories. The following chapters used this point of view. The chapter 5 studied the deformations of thin rods, with a theory more rigorous than the one of Kirchhoff, and the next chapter studied the vibrations of plane membranes. Mathieu recalled that, in his first volume, he studied the vibrations of circular and elliptical plane membranes (*Ivi*, 200). The chapter 7 is devoted to acoustics, with the study of propagation of sound. The chapter 8 explained the vibrations of a curved strip, according to an article of Mathieu in 1882 in the *Journal de l'Ecole Polytechnique*, and the next one, according to the same paper, the vibrations of bells. The final chapter 10 expressed the equilibrium of elasticity of a rectangular prism.

6 Conclusion

In the Book III of his *Optiks*, Newton suggested that, as it is the case for gravity, magnetism, and electricity, all phenomena in Nature could be explained by various attractions of particles of bodies (Newton, 453). Following him, in the *Exposition of the System of World* of 1796, Laplace began a general study of the main results of the application of analysis to phenomena provoked by molecular actions, which are different from the gravity.In the beginning of the XIXth century, in the Arcueil's circle organized by Laplace and Berthollet, this mechanico–molecular view was the first step towards a mathematisation of all the Physics. The first important study in this style was the study of capillarity by Laplace in 1806.

In the hands of Poisson, the attractions are replaced by their potentials, but for him the potentials are only mathematical artefacts. Afterwards, the potential became more concrete and considered as direct representations of physical phenomena. For Lamé, and also for Mathieu, a physical phenomenon is precisely a potential function in a given body, with some bounding conditions and satisfying a specific partial differential equation. Then, the initial analysis by forces between particles loose of its importance, and played a more heuristic part in order to get the potential. Even, as shown by Mathieu, a phenomenon as elasticity could not be explained by mutual attractions of particles alone depending on distances. With Lamé, the Mathematical Physics changed on two points. The physical one is the conception of an universal part play by the aether in any phenomena, the mathematical one is the calculus of curvilinear coordinates. But these two points are linked, when the differential equation for the density of aether at equilibrium is obtained.

The thinking of an inverse potential problem, as introduced and studied by Green, was not truly used in the Lamé's works, but it became central in the Mathieu's contributions. Mathieu took care of the works of his predecessors, by using their phenomenological analysis to get a potential and a differential equation. Then, with a very appropriate mathematical manner, he succeeded to solve the mathematical problem, and he finished by coming back to the experiments and the applications. Several among its contributions are precisely integral representations by calculus of a Green's function.

Whatever these differences, the Courses of Poisson, Lamé and Mathieu are three steps in a large project of an universal method to solve physical problems, and where each theory is compound of a well structured general principles and an open list of particular problems.

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