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Contractible Exact Squares

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In homage to my friend George Janelidze, for his 60th birthday

Abstract. Exact squares in Cat are not necessarily *absolute* (i.e. preserved by any 2-functor Cat $\to$ Cat), or even preserved by any 2-functor given by exponentiation $(-)^{\mathcal{I}}$: Cat $\to$ Cat: if a square is preserved by exponentiation it will be called a *contractible exact square*. We will characterize diagrammatically these contractible squares, and among them the contractible categories, and the so called fibering and cofibering squares, with especially the comma squares and the adjunction squares. As an application we conclude with a diagrammatical characterization of absolutely absolute Kan extensions and especially of absolutely final functors and of absolutely absolute colimits.

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1. Adjunctions, comma squares, distributors, exact squares

In order to be used in the following sections, here are revisited classical notions and notations of "categorical theory of categories", to which we add $(-)^r$, $(-)^l$ and J. We do not explain the deep part played by limits and Kan extensions — everywhere, included in the question of exact squares —, only because in this given paper these facts are not used.

1.1. Adjunctions in Cat and reversions in the double category of 2-squares

In this section I am basically recalling a part of the so-called "mate calculus" the classic reference for which is [20].

1.1.1. Adjunctions. The notion of an adjunction is introduced in 1957 by Kan [19]. Here we recall the definition and we use it for the *reversions* $(-)^r$ and $(-)^l$ of 2-squares.

Definition 1.1. Two functors $U : \mathcal{B} \longrightarrow \mathcal{A}$ and $F : \mathcal{A} \longrightarrow \mathcal{B}$ are *adjoints* (F on the left, and U on the right), and we write $F \dashv U(\epsilon, \eta)$, if for any A object of

$\mathcal{A}$ and B object of $\mathcal{B}$ we have two arrows $\eta_A : A \to UFA$ and $\epsilon_B : FUB \to B$, such that

$$a : A \to UB \mapsto a^\sharp = \epsilon_B.F(a) : FA \to B$$

and

$$b : FA \to B \mapsto b^\flat = U(b).\eta_A : A \to UB$$

are two inverse bijections. And we say that we have natural bijections

$$(-)^\sharp : \mathcal{A}(A, UB) \simeq \mathcal{B}(FA, B) : (-)^\flat$$

It is equivalent to say that there are two natural transformations

$$\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow U.F, \quad \epsilon : F.U \Rightarrow \mathrm{Id}_{\mathcal{B}},$$

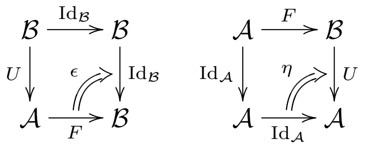
with the two equations

$$(\epsilon.F)(F.\eta) = \mathrm{Id}_F, \quad (U.\epsilon)(\eta.U) = \mathrm{Id}_U.$$

It could be useful to denote η and ϵ by:

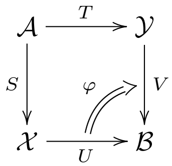
$$\eta = \eta^{F \dashv U} = \eta^{F\dashv} = \eta^{\dashv U} \text{ and } \epsilon = \epsilon^{F\dashv U} = \epsilon^{F\dashv} = \epsilon^{\dashv U}.$$

Agreement: We denote also by the letters ϵ and η the following two 2-squares (according to this notion in the next definition 1.2):



1.1.2. Reversions of 2-squares.

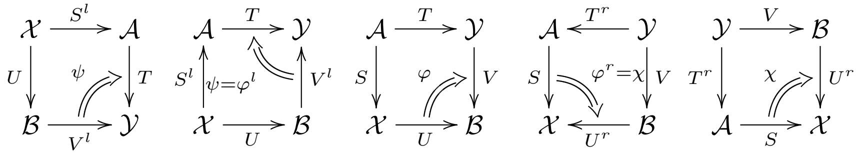
Definition 1.2. A 2-*square* (or a *quintet* in Ehresmann's terminology [6][7]) is the datum in the 2-category Cat of categories, functors and natural transformations, of a picture — denoted in line by $\varphi : US \Rightarrow VT : \mathcal{A}_{\mathcal{X}} \diamond {}^{\mathcal{Y}}\mathcal{B}$, or shortly by $\varphi : US \Rightarrow VT$:



These 2-squares could be composed horizontally and vertically, and so, according to Ehresmann, they constitute a *double category*.

Proposition 1.3. (r)-reversion*: If T and U have right adjoints T^r and U^r, i.e. if $T \dashv T^r$ and $U \dashv U^r$, then the datum a 2-square $\varphi : US \Rightarrow VT$ is equivalent to the datum of a 2-square $\chi : ST^r \Rightarrow U^rV$, denoted by $\chi = \varphi^r$.*
(l)-reversion*: If S and V have left adjoints S^l and V^l, i.e. if $S^l \dashv S$ and $V^l \dashv V$, then the datum of a 2-square $\varphi : US \Rightarrow VT$ is equivalent to the datum of a 2-square $\psi : V^lU \Rightarrow TS^l$, denoted by $\psi = \varphi^l$.*
When they are defined these two reversions are inverse each one of the other:

$$(\varphi^r)^l = \varphi = (\varphi^l)^r.$$



Proof. If $F \dashv U(\epsilon, \eta)$, for any category $\mathcal{C}$ and functors $A : \mathcal{C} \to \mathcal{A}$ and $B : \mathcal{C} \to \mathcal{B}$ we have a bijection and its inverse:

$$\alpha \mapsto \alpha^\sharp = \epsilon_B.F\alpha : \mathrm{Nat}(A, U.B) \simeq \mathrm{Nat}(F.A, B),$$

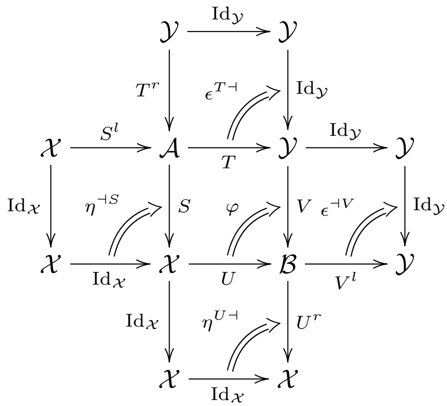
$$\beta \mapsto \beta^\flat = U\beta.\eta_A : \mathrm{Nat}(F.A, B) \simeq \mathrm{Nat}(A, U.B);$$

and dually for any category $\mathcal{D}$ and functors $A' : \mathcal{A} \to \mathcal{D}$ and $B' : \mathcal{B} \to \mathcal{D}$ we have a bijection and its inverse:

$$\alpha' \mapsto {}^\sharp\alpha' = \alpha' F.B'\eta : \mathrm{Nat}(B'.U, A') \simeq \mathrm{Nat}(B', A'.F),$$

$$\beta' \mapsto {}^\flat\beta' = A'\epsilon.\beta' U : \mathrm{Nat}(B', A'.F) \simeq \mathrm{Nat}(B'.U.A, A').$$

Then applying this yoga to the adjunctions $T \dashv T^r$ and $U \dashv U^r$, we get first ${}^\flat\varphi : UST^r \Rightarrow V$, and then $\chi = \varphi^r = ({}^\flat\varphi)^\flat$, and conversely $\varphi = {}^\sharp(\chi^\sharp) = \chi^l$; and then applying to the adjunctions $S^l \dashv S$ and $V^l \dashv V$ we get first $\varphi^\sharp : V^l US \Rightarrow T$, and then $\psi = \varphi^l = {}^\sharp(\varphi^\sharp)$ and conversely $\varphi = ({}^\flat\psi)^\flat = \psi^r$.
We catch a glimpse of the two reversions processes with the diagram:



The horizontal central composition provides $\psi = \varphi^l : V^l U \Rightarrow TS^l$, and the vertical central composition provides $\chi = \varphi^r : S.T^r \Rightarrow U^r V$. $\square$

Remark 1.4. 1 — Of course, if adjoints do exist, then we can continue the process to get $\varphi^{ll} : T^l V^l \Rightarrow S^l U^l$, $\varphi^{rr} : T^r V^r \Rightarrow S^r U^r$, etc.
2 — If $F \dashv U(\epsilon, \eta)$, with identities 2-squares (which of course are invertible as 2-cells) $\delta_U = (\mathrm{Id}_U : \mathrm{Id}_{\mathcal{A}} U \Rightarrow \mathrm{Id}_{\mathcal{A}} U), \gamma_U = (\mathrm{Id}_U : U\mathrm{Id}_{\mathcal{B}} \Rightarrow U\mathrm{Id}_{\mathcal{B}})$ associated to U, and $\delta_F = (\mathrm{Id}_F : \mathrm{Id}_{\mathcal{B}} F \Rightarrow \mathrm{Id}_{\mathcal{B}} F), \gamma_F = (\mathrm{Id}_F : F\mathrm{Id}_{\mathcal{A}} \Rightarrow F\mathrm{Id}_{\mathcal{A}})$ associated to F, we get $\eta : \mathrm{Id}_{\mathcal{A}}\, \mathrm{Id}_{\mathcal{A}} \Rightarrow UF$ and $\epsilon : FU \Rightarrow \mathrm{Id}_{\mathcal{B}}$ as reverse of invertible 2-squares:

$$\eta = \delta_U^l = \gamma_F^r, \qquad \epsilon = \gamma_U^l = \delta_F^r.$$

Furthermore we get:

$$\eta^{ll} = \epsilon, \qquad \epsilon^{rr} = \eta.$$

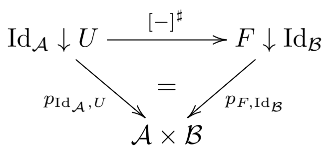
3 — The notion of adjoints and the corresponding calculus of reversions are available in any 2-category or even any bicategory. Especially we will use it in section 1.4 in the case of the bicategory Dist introduced in 1.3: there exact squares will appear as "virtual" left reversions of invertible 2-squares.

1.2. Comma squares, roofs, and the indicator functor J_φ of a 2-square φ

1.2.1. Comma squares or co-junction squares. In [9] Gray proposed a systematic presentation of the calculus of comma categories. Lawvere [21] had introduced the notion of a comma category in 1963, as a generalization of slice and co-slice categories of Grothendieck, in order to express the notion of an adjunction without any reference to sets, in an "elementary" categorical way: if for a comma square (definition 1.5) we introduce the projection

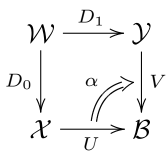
$$p_{U,V} = [D_0, D_1] : U \downarrow V \to \mathcal{X} \times \mathcal{Y},$$

then the fact that $F \dashv U(\epsilon, \eta)$ is equivalent to the existence of an isomorphism $[-]^\sharp : \mathrm{Id}_\mathcal{A} \downarrow U \to F \downarrow \mathrm{Id}_\mathcal{B}$ (with inverse $[-]^\flat : F \downarrow \mathrm{Id}_\mathcal{B} \to \mathrm{Id}_\mathcal{A} \downarrow U$) from $p_{\mathrm{Id}_\mathcal{A},U}$ to $p_{F,\mathrm{Id}_\mathcal{B}}$, i.e. with: $p_{\mathrm{Id}_\mathcal{A},U} = p_{F,\mathrm{Id}_\mathcal{B}}[-]^\sharp$:



So here we recall how a comma square is determined by a co-span, we observe that a "roof" could be determined over a span, and these two facts allow us to associate an *indicator* functor $J_\varphi : S\nabla T \to U \downarrow V$ to a 2-square φ.

Definition 1.5. Given a *co-span* from $\mathcal{X}$ to $\mathcal{Y}$, i.e. two functors $U : \mathcal{X} \to \mathcal{B}$ and $V : \mathcal{Y} \to \mathcal{B}$, we say that a 2-square $\alpha : UD_0 \Rightarrow VD_1 : \mathcal{W}_\mathcal{X} \diamond^\mathcal{Y} \mathcal{B}$



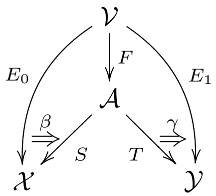
is a *comma* square from U to V if for any 2-square $\varphi : US \Rightarrow VT$ there is a unique $I_\varphi : \mathcal{A} \to \mathcal{W}$ such that

$$D_0 I_\varphi = S, \quad D_1 I_\varphi = T, \quad \alpha I_\varphi = \varphi.$$

Given U and V, the category $\mathcal{W}$ is unique up to isomorphism, and denoted by $\mathcal{W} = U \downarrow V$, and $I_\varphi : \mathcal{A} \to U \downarrow V$, and $\alpha = \alpha_{U,V}$.The old notation was (U, V), hence the name of "comma" category. Perhaps a better name for $U \downarrow V$ would be *co-junction* from U to V.

1.2.2. Roof, attic, ceiling.

Definition 1.6. Given a *span* oriented from $\mathcal{X}$ to $\mathcal{Y}$, i.e. the datum of two functors $S : \mathcal{A} \to \mathcal{X}$ and $T : \mathcal{A} \to \mathcal{Y}$, we define a *roof* over this span as an universal datum $[\mathcal{V}; \beta : E_0 \Rightarrow SF; \gamma : TF \Rightarrow E_1]$ as in the diagram:



Given S and T, the category $\mathcal{V}$ is unique up to isomorphism, and denoted by $\mathcal{V} = S\nabla T$.

So given $[\mathcal{V}'; \beta' : E_0' \Rightarrow SF; \gamma' : TF' \Rightarrow E_1']$, there is a unique $K : \mathcal{V}' \to \mathcal{V}$ such that

$$E_0' = E_0 K, \;\; F' = FK, E_1' = E_1 K, \;\; \beta' = \beta K, \;\; \gamma' = \gamma K.$$

If necessary this K is more explicitly denoted by $K = K_{\beta',\gamma'}$. Such a functor — , under the roof $S\nabla T$ and over the ceiling $H_{S,T}$, hereafter defined. — is thought as being in the *attic* of (S, T), i.e. an element of the functor

$$\mathbb{A}_{S,T} = \mathrm{Cat}(-, S\nabla T).$$

Specially for $[\mathcal{A}; \mathrm{Id}_S : S \Rightarrow S\,\mathrm{Id}_\mathcal{A}; \mathrm{Id}_T : T\,\mathrm{Id}_\mathcal{A} \Rightarrow T]$ we get a factorization named the *ceiling* of (S, T) and denoted by $H_{S,T} = K_{\mathrm{Id}_S,\mathrm{Id}_T} : \mathcal{A} \to S\nabla T$ such that

$$S = E_0 H_{S,T}, \;\; \mathrm{Id}_\mathcal{A} = FH_{S,T}, \;\; T = E_1 H_{S,T}, \;\; \mathrm{Id}_S = \beta H_{S,T}, \;\; \mathrm{Id}_T = \gamma H_{S,T}.$$

1.2.3. Indicator of exactness J_φ.

Proposition 1.7. *Given a span* $(S : \mathcal{A} \to \mathcal{X}; T : \mathcal{A} \to \mathcal{Y})$ *and a co-span* $(U : \mathcal{X} \to \mathcal{B}; V : \mathcal{Y} \to \mathcal{B})$ *in* Cat*, we can construct* $S\nabla T$ *and* $U \downarrow V$*. Furthermore, the data of a natural transformation* $\varphi : US \Rightarrow VT$ *determines a unique functor*

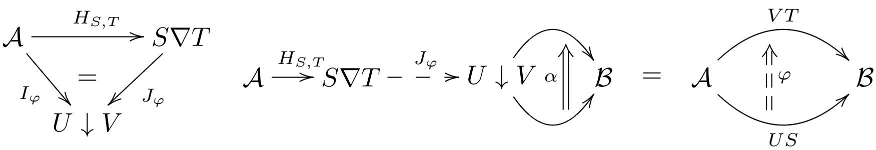
$$J_\varphi : S\nabla T \to U \downarrow V,$$

such that, with the notations of 1.5 *and* 1.6*:*

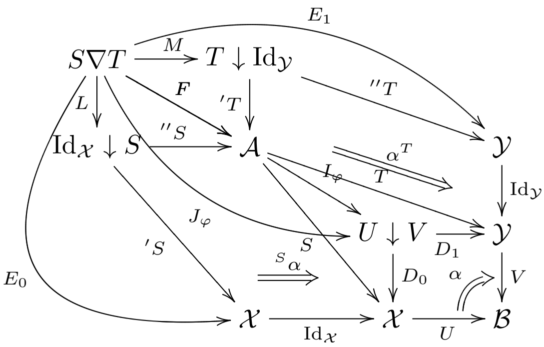
$$D_0 J_\varphi = E_0, \;\; D_1 J_\varphi = E_1, \;\; \alpha J_\varphi = (V\gamma)(\varphi F)(U\beta).$$

This J_φ *will be named the* indicator (of exactness) *of* φ*, and it allows to recover* φ *itself by*

$$J_\varphi H_{S,T} = I_\varphi, \;\; \alpha J_\varphi H_{S,T} = \alpha I_\varphi = \varphi.$$



Proof. As it is known the construction of $U \downarrow V$ in Cat is possible as a projective limit. Then to construct $S\nabla T$ we construct the comma categories $\mathrm{Id}_{\mathcal{X}} \downarrow S$ and $T \downarrow \mathrm{Id}_{\mathcal{Y}}$ with functors $''S$ and $'T$ to $\mathcal{A}$, and $S\nabla T$ is the fibered product $(\mathrm{Id}_{\mathcal{X}} \downarrow S) \times_{\mathcal{A}} (T \downarrow \mathrm{Id}_{\mathcal{Y}})$ of $''S$ and $'T$ over $\mathcal{A}$ as in the diagram:



We define $E_0 = {'SL}$, $F = {''SL} = {'TM}$, $E_1 = {''TM}$, $\beta = {^S\alpha L}$, $\gamma = \alpha^T M$. Given φ, the unicity of J_φ is clear, from the universal property of $U \downarrow V$. And for the existence we just have to know what is an element of $S\nabla T$: it is a datum

$$j = (X; m : X \to SA; A; n : TA \to Y; Y),$$

and we can define

$$J_\varphi(j) = (X; V(n).\varphi_A.U(m) : UX \to VY; Y).$$

Then we have

$$H_{S,T}(A) = (SA; \mathrm{Id}_{SA}; A; \mathrm{Id}_{TA}; TA),\ \ J_\varphi(H_{S,T}(A)) = (SA; \varphi_A; TA) = I_\varphi(A).$$

□

Remark 1.8. 1 — Clearly $S\nabla T = (\mathrm{Id}_{\mathcal{X}} \downarrow S) \times_{\mathcal{A}} (T \downarrow \mathrm{Id}_{\mathcal{Y}})$ is the free 2-sided fibration on the span (S, T) (see this monad in [28]), and by this universal property we get J_φ as the extension of I_φ to a map of fibrations.
2 — Given φ we could have chosen I_φ (see in definition 1.5) to determine φ by a functor. This I_φ $(= J_\varphi H_{S,T})$ will be used in special questions, for instance in the study of fibering and cofibering squares (see section 4). But this is not the functorial representation of φ that we need for the general study of exactness and contractible exactness. The convenient tool in fact is J_φ, because $J_\varphi(j)$ do carry out the composition in $\mathcal{B}$ of the ingredients of j, the result being an element of $U \downarrow V$. We will think of J_φ as an *indicator of exactness* of φ: this will become obvious in section 1.4.

1.2.4. The bifibration $S \boxtimes T$ and the comparison $Q_{S,T} : S\nabla T \to S \boxtimes T$.

Definition 1.9. Given a span $(S : \mathcal{A} \to \mathcal{X}; T : \mathcal{A} \to \mathcal{Y})$ the associated bifibration is the category $S \boxtimes T$ given as "the comma of the co-comma" of S and T, i.e. $S \boxtimes T = G_0 \downarrow G_1$, the comma category (or co-junction) of G_0 and G_1, as in $\alpha_{G_0,G_1} : G_0C_0 \Rightarrow G_1C_1$, where G_0 and G_1 are given by the *co-comma* construction (or *junction* construction) — dual of the comma construction for co-span introduced in definition 1.5— i.e. the co-universal 2-square $\omega_{S,T} : G_0 S \Rightarrow G_1 T$:

$$S \boxtimes T \quad = \quad \begin{array}{ccc} G_0 \downarrow G_1 & \xrightarrow{C_1} & \mathcal{Y} \\ {\scriptstyle C_0}\downarrow & \overset{\alpha_{G_0,G_1}}{\Nearrow} & \downarrow{\scriptstyle G_1} \\ \mathcal{X} & \xrightarrow[G_0]{} & S \uparrow T \end{array} \qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{Y} \\ {\scriptstyle S}\downarrow & \overset{\omega_{S,T}}{\Nearrow} & \downarrow{\scriptstyle G_1} \\ \mathcal{X} & \xrightarrow[G_0]{} & S \uparrow T \end{array}$$

Proposition 1.10. *There is a comparison functor $Q_{S,T} : S\nabla T \to S \boxtimes T$ such that*

$$\begin{array}{ccc} S\nabla T & \xrightarrow{Q_{S,T}} & S \boxtimes T \\ & {\scriptstyle [E_0,E_1]}\searrow \quad = \quad \swarrow{\scriptstyle [C_0,C_1]} & \\ & \mathcal{X} \times \mathcal{Y} & \end{array}$$

Proof. $Q_{S,T}$ is defined by $\alpha_{G_0,G_1} Q_{S,T} = \lambda_{S,T}$ with

$$(\lambda_{S,T})_{(X;m:X\to SA;A;n:TA\to Y;Y)} = n.\omega_A.m.$$

□

1.3. Distributors and virtual right reversion

1.3.1. Distributors. Bénabou [1][2] introduced the notion of bicategory and the calculus of distributors as a basical example of a bicategory.

Definition 1.11. According to [1][2], a *distributor* $\phi : \mathcal{A} \dashrightarrow \mathcal{B}$ from a category $\mathcal{A}$ to a category $\mathcal{B}$ is a functor

$$\phi : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ens},$$

and a *morphism of distributors* θ from ϕ to ϕ' is a natural transformation $\theta : \phi \Rightarrow \phi'$. So we get a category $\mathrm{Dist}(\mathcal{A}, \mathcal{B}) = \mathrm{Cat}(\mathcal{B}^{\mathrm{op}} \times \mathcal{A}, \mathrm{Ens})$.
With $\Phi(A) = \phi(-, A)$, a distributor $\phi : \mathcal{A} \dashrightarrow \mathcal{B}$ could also be seen as a true functor $\Phi : \mathcal{A} \longrightarrow \mathrm{Ens}^{\mathcal{B}^{\mathrm{op}}}$, which is a generalization of a functor $F : \mathcal{A} \to \mathcal{B}$.

Let $\phi : \mathcal{A} \dashrightarrow \mathcal{B}$ and $\psi : \mathcal{B} \dashrightarrow \mathcal{C}$ be two consecutive distributors. The composite $\psi \otimes \phi : \mathcal{A} \dashrightarrow \mathcal{C}$ is defined by

$$(\psi \otimes \phi)(C, A) = \Big(\coprod_B \psi(C, B) \times \phi(B, A) \Big) / \equiv,$$

where $\equiv$ is the equivalence relation generated by the relation

$$(\psi(C, b)(\beta), \alpha) \sim (\beta, \phi(b, A)(\alpha))$$

for $\alpha \in \phi(B, A)$, $\beta \in \psi(C, B')$ and $b \in \mathcal{B}(B', B)$.
So, if we denote by $y \otimes x$ the equivalence class $(y, x) \mod \equiv$, we have

$$b\beta \otimes \alpha = \beta \otimes \alpha b.$$

Then given $\theta : \phi \Rightarrow \phi'$ and $\tau : \psi \Rightarrow \psi'$, we define

$$\psi' \otimes \theta : \psi' \otimes \phi \Rightarrow \psi' \otimes \phi', \quad \tau \otimes \phi : \psi \otimes \phi \Rightarrow \psi' \otimes \phi,$$

by

$$(\psi' \otimes \theta)_{(C,A)}(y' \otimes x) = y' \otimes \theta_{(B,A)}(x), \quad (\tau \otimes \phi)_{(C,A)}(y \otimes x) = \tau_{(C,B)}(y) \otimes x,$$

where $x \in \phi(B,A)$ and $y \in \psi(C,B)$, $y' \in \psi'(C,B)$ and then

$$\tau \otimes \theta = (\psi' \otimes \theta).(\tau \otimes \phi) = (\tau \otimes \phi').(\psi \otimes \theta),$$

where . denotes the composition of natural transformations, and so

$$(\tau \otimes \theta)_{(C,A)}(y \otimes x) = \tau_{(C,B)}(y) \otimes \theta_{(B,A)}(x).$$

With categories as objects or 0-cells, distributors as 1-cells, and morphisms of distributors as 2-cells, with this composition $\otimes$, with some precise coherences morphisms, Bénabou ([1],[2]) has shown that we get a *bicategory* denoted by Dist. He also shows that to any functor $F : \mathcal{A} \to \mathcal{B}$ are associated two distributors:

- $\phi_F : \mathcal{A} \dashrightarrow \mathcal{B} : (B,A) \mapsto \mathcal{B}(B, FA)$,
- $\phi^F : \mathcal{B} \dashrightarrow \mathcal{A} : (A,B) \mapsto \mathcal{B}(FA, B)$,

and the association

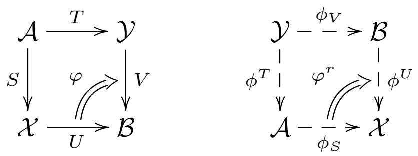
$$\phi_{(-)} : \mathrm{Cat} \longrightarrow \mathrm{Dist} : F \mapsto \phi_F$$

is an embedding of the 2-category Cat in the bicategory Dist, such that for any $F : \mathcal{A} \to \mathcal{B}$ we have in Dist the adjunction

$$\phi_F \dashv \phi^F.$$

1.3.2. Virtual right reversion.

Proposition 1.12. *A 2-square $\varphi : US \Rightarrow VT : \mathcal{A}_{\mathcal{X}} \diamond {}^{\mathcal{Y}}\mathcal{B}$ in* Cat *has a* virtual right reversion *in* Dist*: $\varphi^r := (\phi_\varphi)^r : \phi_S \otimes \phi^T \Rightarrow \phi^U \otimes \phi_V : \mathcal{Y}_{\mathcal{A}} \diamond {}^{\mathcal{B}}\mathcal{X}$:*



and this φ^r is directly determined by J_φ.

Proof. This is an immediate consequence of $\phi_F \dashv \Phi^F$ in Dist, with the yoga of reversions as introduced in Cat in Proposition 1.3, but of course completely similar in Dist. We have only to precise that

$$(\phi_s \otimes \phi^T)(X,Y) = \{j = [m,n] = (X; m : X \to SA; A; n : TA \to Y; Y)\}/\equiv,$$

$$\phi^U \otimes \phi_V(X,Y) = \mathcal{B}(UX, VY),$$

in such a way that, with $n \otimes m = j \mod \equiv$:

$$\varphi^r_{X,Y} : (\phi_S \otimes \phi^T)(X,Y) \to \phi^U \otimes \phi_V(X,Y) : n \otimes m \mapsto V(n)\varphi_A.U(m) = J_\varphi(j),$$

with $J_\varphi(j)$ so defined in Proposition 1.7. $\square$

1.4. Exact squares

1.4.1. Direct definition of an exact 2-square via distributors. The calculus of exact squares had been introduced by Guitart [11], as simultaneously a generalization of absolute colimits (and absolute Kan extensions) in Cat and a generalization of exact sequences (and exact squares) in Ab. With the help of previous sections, we are ready to express the definition of an exact square via distributors:

Definition 1.13. An *exact square* in Cat is a 2-square $\varphi : US \Rightarrow VT : \mathcal{A}_{\mathcal{X}} \diamond^{\mathcal{Y}} \mathcal{B}$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{Y} \\ {\scriptstyle S}\downarrow & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle V} \\ \mathcal{X} & \xrightarrow[U]{} & \mathcal{B} \end{array}$$

such that its virtual right reversion φ^r is invertible.

Proposition 1.14. *If the functor U and T admit right adjoints, in such a way that $\varphi^r : ST^r \Rightarrow U^rT$ is a well-defined natural transformation, then the exactness is equivalent to the Beck-Chevalley condition*

$$\varphi^r : ST^r \simeq U^rT.$$

Proposition 1.15. *Any adjunction 2-square η or ϵ (as it is agreed in Definition* 1.1*) is exact, and in fact the exactness of η as well as the exactness of ϵ characterizes the fact that we have an adjunction.*

Proof. A proof is given in [11]. This proposition will be strenghtened in Proposition 4.4. □

A more elementary characterization of exactness given in ([11, prop 1.4]) will be used in Proposition 4.7:

Proposition 1.16. *The 2-square $\varphi : US \Rightarrow VT$ is exact if and only for any category $\mathcal{Z}$ and functors $P : \mathcal{X} \to \mathcal{Z}$ and $Q : \mathcal{Y} \to \mathcal{Z}$, the map*

$$(-)I_\varphi : \mathrm{Nat}(PD_0, QD_1) \to \mathrm{Nat}(PS, QT) : \theta \mapsto \theta I_\varphi$$

is bijective. Especially any comma square is exact.

Remark 1.17. 1 — The definition and the three previous propositions are in [11], with other characterizations, related to localization or to preservation of Kan extensions, related to "multiplicative squares", with examples as fully faithful functor, final functor, opaque functor (see here below Definition 5.9), etc.
2 — For example if $\mathcal{Y} = \mathbf{1}$ the final category, and if $\mathcal{A}$, $\mathcal{X}$ and $\mathcal{B}$ are groups, and with φ the equality, then the square is exact if and only if U is surjective and $\mathrm{im}S = \ker U$ (proof by application of criterion 1 in Proposition 1.18), i.e. the sequence of groups $\mathcal{A} \to \mathcal{X} \to \mathcal{B} \to \mathbf{1}$ is exact.
3 — Applications to logic and calculus of relations are given in [11] and [12], applications to homological algebra and satellites are given in [14] [15] and [13], and applications to shape theory are given in [11] and after in [4] and

[5]. Many of the results of [11] are reproduced in [5]. Other aspects, as link with calculus of spans of functions, with Yoneda-structures, and with analysis of difunctional relations, are developped in [30], [31], [26]. More recent applications in cohomology are given in [18] and [22].
4 — For the needs of this paper, and the characterization of contractible exact squares now we emphasize the characterization of exactness by properties of J_φ (Proposition 1.18) or of I_φ, and by the examples of comma squares and adjunction 2-squares as in Proposition 1.16 and Proposition 1.15.

1.4.2. Elementary characterization of an exact 2-square via J_φ. If we make explicit the Definition 1.13, with the help of Proposition 1.12 and Proposition 1.7, we easily verify that (proof also in [11], Proposition 1.2 [Zig-zag criterion]):

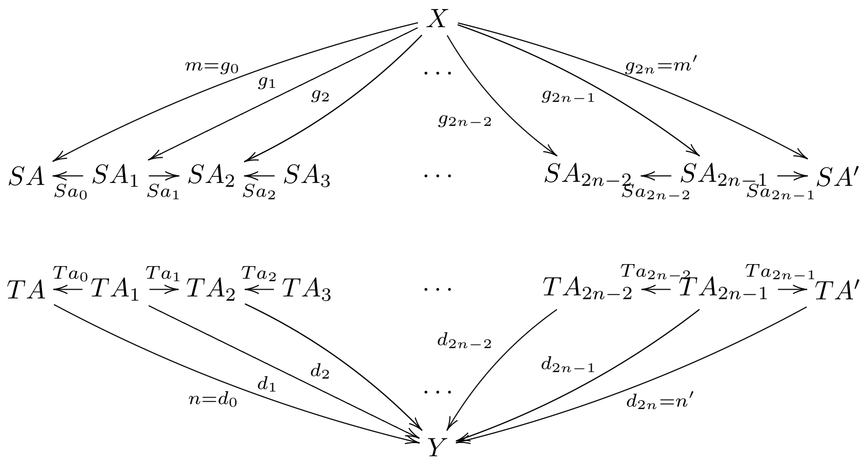
Proposition 1.18. *1 — The 2-square $\varphi : US \Rightarrow VT$ is exact if and only if the two following conditions are satisfied:*
(1) Given any objects X in $\mathcal{X}$ and Y in $\mathcal{Y}$ and any map $b : UX \to VY$ — that is to say given an object $w = (X; b : UX \to VY; Y)$ in $U \downarrow V$ — there is a solution to the problem of ϕ-factorization of b, i.e. there is an object A in $\mathcal{A}$ and two maps $m : X \to SA$ in $\mathcal{X}$ and $n : TA \to Y$ in $\mathcal{Y}$ such that

$$Vn.\phi_A.Um = b.$$

(2) Given (m, A, n) a solution of (1) as above, and (m', A', n') another solution — with $Vn'.\phi_{A'}.Um' = b$ —, then these two solutions are connected, i.e. there is a zig-zag in $\mathcal{A}$ from A to A',

$$A \xleftarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xleftarrow{a_2} A_3 \ldots\ldots A_{2n-2} \xleftarrow{a_{2n-2}} A_{(2n-1)} \xrightarrow{a_{2n-1}} A'$$

and a lantern:



where any triangle commutes: $m = g_0 = Sa_0.g_1, m' = g_{2n} = Sa_{2n-1}.g_{2n-1}$, and $g_{2k} = Sa_{2k-1}.g_{2k-1} = Sa_{2k}.g_{2k+1}$, for $1 \leq k \leq n-1$; and $n = d_0$, $d_{2n} = n'$, and $d_{2k+1} = d_{2k}.Ta_{2k} = d_{2k+2}.Ta_{2k+1}$, for $0 \leq k \leq n-1$.

2 — The 2*-square* $\varphi : US \Rightarrow VT$ *is exact if and only for every objet* $w \in U \downarrow V$ *the category* $J_\varphi^{-1}(w)$ *is connected non-empty, i.e.* $\pi_0(J_\varphi^{-1}(w)) = 1$*.*

3 — The 2*-square* $\varphi : US \Rightarrow VT$ *is exact if and only the functor* J_φ *factorizes through* $Q_{S,T}$ *by an isomorphism:*

$$\begin{array}{ccc} S\nabla T & \xrightarrow{Q_{S,T}} & S \boxtimes T \\ & \underset{J_\varphi}{\searrow} \;=\; \underset{\simeq}{\swarrow} & \\ & U \downarrow V & \end{array}$$

2. Contractible exact squares: definition and zig-zag criteria

Here we introduce the central notion of this paper.

Definition 2.1. We define a *contractible exact square* in Cat as a 2-square $\varphi : US \Rightarrow VT : \mathcal{A}_{\mathcal{X}} \diamond^{\mathcal{Y}} \mathcal{B}$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{Y} \\ {\scriptstyle S}\downarrow & \overset{\varphi}{\Nearrow} & \downarrow{\scriptstyle V} \\ \mathcal{X} & \xrightarrow[U]{} & \mathcal{B} \end{array}$$

such that for any category $\mathcal{I}$, the 2-square $\varphi^{\mathcal{I}} : U^{\mathcal{I}} S^{\mathcal{I}} \Rightarrow V^{\mathcal{I}} T^{\mathcal{I}} : \mathcal{A}^{\mathcal{I}}{}_{\mathcal{X}^{\mathcal{I}}} \diamond^{\mathcal{Y}^{\mathcal{I}}} \mathcal{B}^{\mathcal{I}}$

$$\begin{array}{ccc} \mathcal{A}^{\mathcal{I}} & \xrightarrow{T^{\mathcal{I}}} & \mathcal{Y}^{\mathcal{I}} \\ {\scriptstyle S^{\mathcal{I}}}\downarrow & \overset{\varphi^{\mathcal{I}}}{\Nearrow} & \downarrow{\scriptstyle V^{\mathcal{I}}} \\ \mathcal{X}^{\mathcal{I}} & \xrightarrow[U^{\mathcal{I}}]{} & \mathcal{B}^{\mathcal{I}} \end{array}$$

— where $S^{\mathcal{I}}(W) = SW$, etc., $(\varphi^{\mathcal{I}})_W = \varphi W$ — is exact i.e. (see Definition 1.13) such that its virtual right reversion $(\varphi^{\mathcal{I}})^r$ is invertible.

Proposition 2.2. *A* 2*-square* $\varphi : US \Rightarrow VT$ *is a* contractible exact square *if and only if there is a functor* $K : U \downarrow V \to S\nabla T$ *such that* $J_\varphi K = \mathrm{Id}_{U\downarrow V}$*, a sequence of functors* $R_q : S\nabla T \to S\nabla T$*, with* $0 \le q \le 2n$*, and in* $(S\nabla T)^{S\nabla T}$ *a zig-zag*

$$\mathrm{Id}_{S\nabla T} = R_0 \xleftarrow{\theta_0} R_1 \xrightarrow{\theta_1} R_2 \ldots R_{2n-2} \xleftarrow{\theta_{2n-2}} R_{2n-1} \xrightarrow{\theta_{2n-1}} R_{2n} = KJ_\varphi,$$

with

$$J_\varphi \theta_q = \mathrm{Id}_{J_\varphi}.$$

Proof. As $(-)^{\mathcal{I}}$ is a right adjoint and $((-)^{\mathcal{I}})^{\mathbf{2}} \simeq ((-)^{\mathbf{2}})^{\mathcal{I}}$ (with **2** the category with two objects 0 and 1, and one non-identity arrow, from 0 to 1), we have

$$U^{\mathcal{I}} \downarrow V^{\mathcal{I}} \simeq (U \downarrow V)^{\mathcal{I}}, \quad S^{\mathcal{I}} \nabla T^{\mathcal{I}} \simeq (S\nabla T)^{\mathcal{I}}, \quad J_{\varphi^{\mathcal{I}}} \simeq (J_\varphi)^{\mathcal{I}}.$$

So if we assume that the square is contractible exact, for $\mathcal{I} = U \downarrow V$ we get a K such that $(J_\varphi)^{\mathcal{I}}(K) = \mathrm{Id}_{U\downarrow V}$, and then for $\mathcal{I} = S\nabla T$ we have

$(J_\varphi)^{S\nabla T}(K.J_\varphi) = (J_\varphi)^{S\nabla T}(\mathrm{Id}_{S\nabla T})$ and so $\mathrm{Id}_{S\nabla T}$ and KJ_φ have to be connected over J_φ.
Conversely, with the given condition, if $M : \mathcal{I} \to U \downarrow V$, then $J_\varphi(KM) = M$. If $N : \mathcal{I} \to S\nabla T$ and $N' : \mathcal{I} \to S\nabla T$ are such that $J_\varphi N = J_\varphi N'$, then over J_φ we have a zig-zag:

$$N \longleftarrow R_1 N \ldots R_{2n-1} N \longrightarrow KJ_\varphi N = KJ_\varphi N' \longleftarrow R_{2n-1} N' \ldots R_1 N' \longrightarrow N'.$$

□

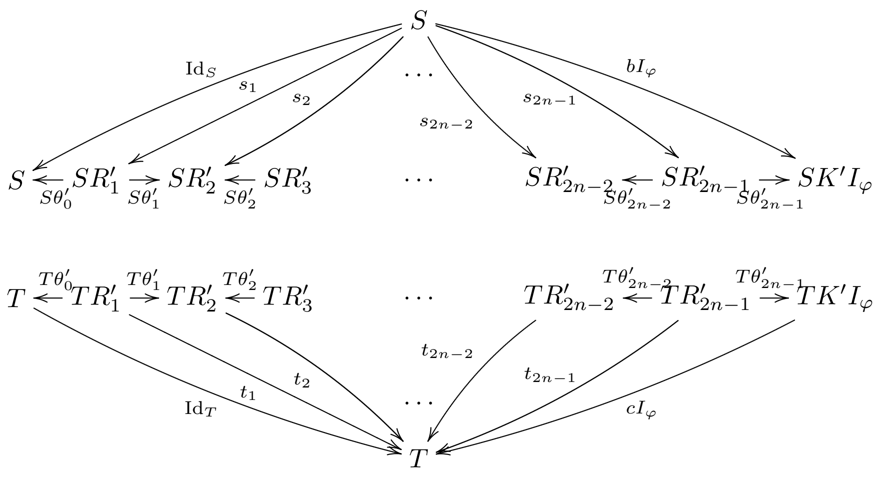
Proposition 2.3. *A 2-square $\varphi : US \Rightarrow VT$ is a* contractible exact square *if and only if there is a functor $K' : U \downarrow V \to \mathcal{A}$ and natural transformations $b : D_0 \to SK'$ and $c : TK' \to D_1$ such that*

$$(Vc)(\varphi K')(Ub) = \alpha,$$

and a sequence of functors $R'_q : \mathcal{A} \to \mathcal{A}$, with $0 \le q \le 2n$, and in $\mathcal{A}^{\mathcal{A}}$ a zig-zag

$$\mathrm{Id}_{\mathcal{A}} = R'_0 \xleftarrow{\theta'_0} R'_1 \xrightarrow{\theta'_1} R'_2 \ldots R'_{2n-2} \xleftarrow{\theta'_{2n-2}} R'_{2n-1} \xrightarrow{\theta'_{2n-1}} R'_{2n} = K' I_\varphi,$$

and a lantern



Proof. For $\mathcal{I} = U \downarrow V$ contractible exactness implies, by the zig-zag condition in Proposition 2.2, the existence of K', b and c; then for $\mathcal{I} = \mathcal{A}$, contractible exactness implies that $(S; \mathrm{Id}_S; \mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_T; T)$ and $(S; bI_\varphi; K'I_\varphi, cI_\varphi; T)$ are connected over φ.
Conversely, starting with the zig-zag condition in this proposition we get the zig-zag condition in Proposition 2.2 with:

$$R_q(x; \beta; a, \gamma, y) = (x; (s_q)_a.\beta; R'_q(a); \gamma(t_q)_a.y),$$

$$(\theta_q)_{(x;\beta;a,\gamma,y)} = (\mathrm{Id}_x, (\theta'_q)_a, \mathrm{Id}_y).$$

□

3. Contractible category

As a first interesting special case of a contractible square we get the notion of a *contractible category*. We have to notice that this notion is strictly stronger than the one of contractibility of the classifying space (see Proposition 3.6). It is from this case that we had chosen the name of a "contractible" square for the general situation (Definition 2.1).

Definition 3.1. A *contractible category* $\mathcal{C}$ is a category such that in the category $\mathcal{C}^{\mathcal{C}}$ the identity functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$ is connected by a zig-zag of natural transformations $(\theta_k)_{0\leq k\leq 2n-1}$ to some constant functor $A^\dagger$, with A an object of $\mathcal{C}$ (and such a zig-zag θ is named a *contraction* of $\mathcal{C}$ on A, of length $2n$):

$$\mathrm{Id}_{\mathcal{C}} = T_0 \xleftarrow{\theta_0} T_1 \xrightarrow{\theta_1} T_2 \ldots T_{2n-2} \xleftarrow{\theta_{2n-2}} T_{2n-1} \xrightarrow{\theta_{2n-1}} T_{2n} = A^\dagger \qquad (\star_{2n})$$

In other words, there is an object A in $\mathcal{C}$, an integer n, a functor $\theta : \mathcal{Z}_{2n} \longrightarrow \mathcal{C}^{\mathcal{C}}$ with $\theta(0) = \mathrm{Id}_{\mathcal{C}}$, $\theta(i) = T_i$, and $\theta(2n) = A^\dagger$, and $\theta(z_k) = \theta_k$, where $\mathcal{Z}_{2n}$ is the category :

$$0 \xleftarrow{z_0} 1 \xrightarrow{z_1} 2 \xleftarrow{z_2} 3 \ldots\ldots 2n-2 \xleftarrow{z_{2n-2}} (2n-1) \xrightarrow{z_{2n-1}} 2n \qquad (\mathcal{Z}_{2n})$$

Proposition 3.2. *A category $\mathcal{C}$ is* contractible *if and only $\mathcal{C}$ is non-empty and for any category $\mathcal{I}$ the category $\mathcal{C}^{\mathcal{I}}$ is connected. Furthermore, if $\mathcal{C}$ is contractible, then for any $\mathcal{D}$ the category $\mathcal{C}^{\mathcal{D}}$ is contractible too.*

Proof. By Definition $\mathcal{C}$ is contractible if and only $\mathcal{C}$ is non-empty and $\mathcal{C}^{\mathcal{C}}$ is connected; and if $\mathcal{C}$ is contracted on A by a $(\star)$ and if F and G are two objects of $\mathcal{C}^{\mathcal{I}}$ they are both connected to $A^\dagger$ by the sequences $(\star).F$ and $(\star).G$. The last point comes from the isomorphism $\left(\mathcal{C}^{\mathcal{D}}\right)^{\mathcal{I}} \simeq \mathcal{C}^{\mathcal{D}\times\mathcal{I}}$. □

Proposition 3.3. *A category $\mathcal{C}$ is* contractible *if and only for any $\mathcal{I}$ the 2-square*

$$\begin{array}{ccc} \mathcal{C}^{\mathcal{I}} & \longrightarrow & \mathbf{1} \\ \downarrow & = & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbf{1} \end{array}$$

is exact; and for that it is enough $\mathcal{C}$ be non-empty and have this property in the case $\mathcal{I} = \mathcal{C}$.

Proof. A category $\mathcal{C}$ is connected and non-empty if and only if the 2-square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathbf{1} \\ \downarrow & = & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbf{1} \end{array}$$

is exact, this is clear from the zig-zag criterion in Proposition 1.18. Then the zig-zag criterion in Proposition 2.3 and Proposition 3.2 provide exactly the contractibility as in Definition 3.1, and the equivalent property that the 2-square above is a contractible exact square.

□

Proposition 3.4. *A category $\mathcal{C}$ could be connected and non-empty yet not contractible. It is the case for the category $\mathbf{2}_2$ with two object* 0 *and* 1 *and two non-identity arrows* 2_1 *and* 2_2*, from* 0 *to* 1.

Proof. We see directly that $\mathbf{2}_2^{\mathbf{2}_2}$ is not connected, because there the object $\mathrm{Id}_{\mathbf{2}_2}$ is isolated i.e. disconnected, without non-identity arrow from or toward it. Otherwise, applying the first part of Proposition 3.6, we get that $\mathbf{2}_2$ is not contractible because its classifying space, which is the circle, is not a contractible space. □

Proposition 3.5. *For any $n \in \mathbb{N}$ the category $\mathcal{Z}_{2n}$ itself is contractible.*

Proof. A contraction like $(\star_{2n})$ — of length $2n$ exactly — from $\mathrm{Id}_{\mathcal{Z}_{2n}}$ to $0^\dagger$ is given by: $T_i(k) = \inf(k, 2n-i)$, $(\theta_i)_k = \begin{cases} \mathrm{Id}_k & \text{if } k < 2n-i \\ z_{2n-i-1} & \text{if } k \geq 2n-i \end{cases}$. □

Proposition 3.6. *If a category $\mathcal{C}$ is contractible, then its classifying space $\mathrm{B}\mathcal{C} = |N\mathcal{C}|$ — the geometric realization of its nerve — is a contractible space, but the converse is false.*

Proof. A topological space X is *contractible* ("contractile" in french) if the identity map $\mathrm{Id}_X : X \to X : x \mapsto x$ is homotopic to some constant map $a^\dagger : X \to X : x \mapsto a$, with $a \in X$.
Let $\mathbf{2}$ be the category with two objects 0 and 1, and one non-identity arrow, from 0 to 1: we have $\mathrm{B}\mathbf{2} = [0,1]$. If $\mathcal{C}$ is contractible in virtue of a zig-zag $(\star)$, then each $T_k : \mathcal{C} \to \mathcal{C}$ generates a continous map $\mathrm{B}T_k : \mathrm{B}\mathcal{C} \to \mathrm{B}\mathcal{C}$, and each of the $\theta_{2q\pm1} : T_{2q\pm1} \to T_{2q}$, as a functor $\Theta_{2q\pm1} : \mathbf{2} \times \mathcal{C} \to \mathcal{C}$ generates a homotopy $[0,1] \times \mathrm{B}\mathcal{C} \simeq \mathrm{B}(\mathbf{2} \times \mathcal{C}) \xrightarrow{\mathrm{B}\Theta_{2q\pm1}} \mathrm{B}\mathcal{C}$ from $\mathrm{B}T_{2q\pm1}$ to $\mathrm{B}T_{2q}$, and by composition of these homotopies we get a homotopy from $\mathrm{BId}_\mathcal{C}$ to $\mathrm{B}A^\dagger$.
And the converse is false, according to the following *counterexample*. Let $\mathcal{Z}_\infty$ be the "infinite zig-zag" starting from an object "0", with objects the integers $n \in \mathbb{N}$ and with morphisms the couples $z_{2k} = (2k, 2k+1) : 2k+1 \to 2k$ and the couples $z_{2k+1} = (2k+2, 2k+1) : 2k+1 \to 2k+2$, for any $k \in \mathbb{N}$:

$$0 \xleftarrow{z_0} 1 \xrightarrow{z_1} 2 \xleftarrow{z_2} 3 \ldots\ldots 2n-2 \xleftarrow{z_{(2n-2)}} (2n-1) \xrightarrow{z_{2n-1}} 2n \xleftarrow{z_{2n}} \ldots \qquad (\mathcal{Z}_\infty)$$

Then $\mathrm{B}z_q = [q, q+1]$, and $\mathrm{B}\mathcal{Z}_\infty = \cup_{q \geq 0}[q, q+1] = [0, \infty[$ as a topological space is contracted on 0 by $h(x,t) = x(1-t)$; but $\mathcal{Z}_\infty$ is not a contractible category, because the minimal zig-zag from m to 0 is of length m, for any m, but it would be $\leq n_0$ if $\mathcal{Z}_\infty$ were contracted by a $(\star)$ of length n_0. □

Proposition 3.7. *A category $\mathcal{C}$ is contractible (def.* 3.1*) if and only if $\mathcal{C} \to \mathbf{1}$ is absolutely final (def.* 5.16*) i.e. for every* 2*-functor* $\Phi : \mathrm{Cat} \to \mathrm{Cat}$*, the functor*

$$\Phi(\mathcal{C}) \to \Phi(\mathbf{1})$$

is final, i.e. the functor

$$\Big(\big(\Phi(\mathcal{C}) \to \Phi(\mathbf{1})\big) \downarrow \Phi(\mathbf{1}) \Big) \to \Phi(\mathbf{1})$$

has non-empty connected fibers.

Proof. By application of Proposition 3.2 or Proposition 3.3, Proposition 5.8, Proposition 5.17. □

4. Fibering and Cofibering squares, liftors and co-liftors

As a second special case of contractible exact squares, we have the *fibering* and the *cofibering squares* (see Proposition 4.5), and more specially, *liftors* and *co-liftors*.

4.1. Fibering and Cofibering squares

Definition 4.1. 1 — A 2-square $\varphi : US \Rightarrow VT : \mathcal{A}_{\mathcal{X}} \diamond^{\mathcal{Y}} \mathcal{B}$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{Y} \\ {\scriptstyle S}\downarrow & \overset{\varphi}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle V} \\ \mathcal{X} & \xrightarrow[U]{} & \mathcal{B} \end{array}$$

is said to be a *fibering square* if the functor $I_\varphi : \mathcal{A} \to U \downarrow V$ has a right adjoint R, i.e. $I\varphi \dashv R(\epsilon, \eta)$, with furthermore the condition that $D_0\epsilon : SR \Rightarrow D_0$ is an isomorphism.

2 — The square $\varphi : US \Rightarrow VT : \mathcal{A}_{\mathcal{X}}$ is a *cofibering square* if the functor $I_\varphi : \mathcal{A} \to U \downarrow V$ has a left adjoint L, i.e. $L\varphi \dashv I_\varphi(\epsilon, \eta)$, with furthermore the condition that $D_1\epsilon : TL \Rightarrow D_1$ is an isomorphism.

This definition (as well as the name of the notion) is given in [15, 14.8, p.367]. It comes by "symmetrization" from an observation [15, 7.1, p.323] which could be expressed by:

Proposition 4.2. *A functor $F : \mathcal{A} \to \mathcal{X}$ is a fibration (resp. a cofibration) in the sense of Grothendieck if and only if the* 2*-square*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}_\mathcal{A}} & \mathcal{A} \\ {\scriptstyle F}\downarrow & \overset{\mathrm{Id}_F}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle F} \\ \mathcal{X} & \xrightarrow[Id_\mathcal{X}]{} & \mathcal{X} \end{array}$$

(which trivially is always exact) is a fibering square (resp. a cofibering square) according to Definition 4.1.

Proof. In fact this observation is known, it was already in Gray [8] and there attributed to Chevalley; and this criterion is used by Street in [27] to define fibrations in 2-categories with enough limits. □

Proposition 4.3. *A comma square is fibering and cofibering.*

Proof. It is obvious, as in this case $I_\varphi = \mathrm{Id}_{U\downarrow V}$. □

Proposition 4.4. *Any adjunction* 2*-square η (or ϵ) (as in Proposition* 1.15*) is a fibering square (resp. a cofibering square), and this characterizes the fact that we have an adjunction.*

Proof. Given an adjunction $F \dashv U(\epsilon, \eta)$, we get $I_\eta : \mathcal{A} \to \mathrm{Id}_\mathcal{A} \downarrow U$ with $I_\eta(A) = (A; \eta_A; FA)$, and its right adjoint $R : \mathrm{Id}_\mathcal{A} \downarrow U \to \mathcal{A}$ given by $R(A, m, B) = A$, and so the square η is fibering. For the reciprocal, with Proposition 4.5 we get the fact that η is contractible exact and so is exact, and, by Proposition 1.15 it is an adjunction square. □

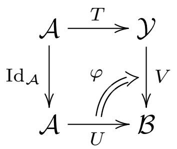
Proposition 4.5. *If a* 2*-square is fibering or cofibering, then it is a contractible exact square.*

Proof. The fact was announced without proof in [15, 14.8, p.368] (where contractible squares are named strong exact squares). Here we give the details, as an application of the criterion in Proposition 2.3.
So let φ be a fibering square, with $I_\varphi \dashv R(\epsilon, \eta)$, and $D_0\epsilon : SR \Rightarrow D_0$ an isomorphism, with $\epsilon : I_\varphi R \Rightarrow \mathrm{Id}_{U\downarrow V}$ and $\eta : \mathcal{A} \Rightarrow RI_\varphi$, with the equations $(\epsilon I_\varphi)(I_\varphi\eta) = \mathrm{Id}_{I_\varphi}$ and $(R\epsilon)(\eta R) = \mathrm{Id}_R$. Here in fact we need only the first one. We note $\mathcal{K} = U \dashrightarrow V$ and $I = I_\varphi$, and we take $b = (D_0\epsilon)^{-1}$ and $c = D_1\epsilon$. As $\alpha = \alpha_{U,V} : UD_0 \Rightarrow VD_1$ is natural we have $\alpha(UD_0\epsilon) = (UD_1\epsilon)(\alpha IR)$ and with $\alpha I = \varphi$, this gives $(Vc)(\varphi R)(Ub) = \alpha$, which is the first condition we need. For the two other conditions, namely $bI = S\eta$ and $(cI)(T\eta) = \mathrm{Id}_T$, we apply to the first adjunction equation the functor D_0 and the functor D_1: we get $(D_0\epsilon I)(D_0 I\eta) = D_0 I$ and $(D_1\epsilon I)(D_0 I\eta) = D_1 I$,, which is the result. □

4.2. Liftors and co-liftors

Definition 4.6. A *liftor* is the special case of a fibering square as in the following Proposition 4.7, where S is an identity. When U is an injection it could be seen as a *partial adjunction*, and of course if U is an identity functor, we just have an η- adjunction square.

Proposition 4.7. *For a* 2*-square* $\varphi : US \Rightarrow VT$ *where* $\mathcal{A} = \mathcal{X}$ *and* $S = \mathrm{Id}_A$



the following conditions are equivalent:
— *The square is exact.*
— *There is a natural transformation* $\chi : TD_0 \Rightarrow D_1$ *such that* $\chi I_\varphi = \mathrm{Id}_T$.
— $I_\varphi \dashv D_0(\mathrm{Id}_\mathcal{A}, \eta)$*, with* $D_0\eta = \mathrm{Id}_{D_0}$.
— *The square is fibering.*
— *The square is contractible exact.*

Proof. If the square is exact, by Proposition 1.16 there is a unique $\chi : TD_0 \to D_1$ such that $\chi T = \mathrm{Id}_T$.
Then $\varphi = \varphi$ is equivalent to $(V\chi I_\varphi)(\varphi D_0 I_\varphi) = \alpha I_\varphi$, or $(V\chi)(\varphi D_0) = \alpha$, or $(V\chi)(\alpha I_\varphi D_0) = \alpha \mathrm{Id}_{D_0}$: so there is a unique $\eta : ID_0 \to \mathrm{Id}_{U\downarrow V}$ such that $D_0\eta = \mathrm{Id}_{D_0}$ and $D_1\eta I_\varphi = \mathrm{Id}_T$. Also we have $D_0 I_\varphi = \mathrm{Id}_\mathcal{A}$, and we have $I \dashv D_0(\mathrm{Id}_\mathcal{A}, \eta)$: $((\eta I_\varphi)(I_\varphi \mathrm{Id}_\mathcal{A}) = \eta I_\varphi = \mathrm{Id}_I$, because of $D_0\eta I_\varphi = \mathrm{Id}_\mathcal{A}$ and

$D_1\eta I_\varphi = \mathrm{Id}_T$; and $(D_1\eta)\mathrm{Id}_{D_0} = \mathrm{Id}_{D_0}$. And so the square is fibering. Then it is contractible exact, by Proposition 4.5.

□

Remark 4.8. In [11] it is observed that a 2-square $\varphi : US \to VT$ determines a *dual* $\varphi^{\mathrm{op}} : V^{\mathrm{op}}T^{\mathrm{op}} \Rightarrow U^{\mathrm{op}}S^{\mathrm{op}}$, and this second square is exact if and only if the first one is exact. So from the previous proposition we get a dual one for the case of a *co-liftor* where $T = \mathrm{Id}_\mathcal{A}$.

As a special case we get:

Proposition 4.9. *Given a functor $U : \mathcal{A} \to \mathcal{B}$ the following properties are equivalent:*
— *U is full and faithful.*
— *The square*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{Id}_\mathcal{A}} & \mathcal{A} \\ {\scriptstyle \mathrm{Id}_\mathcal{A}}\downarrow & = & \downarrow{\scriptstyle U} \\ \mathcal{A} & \xrightarrow[U]{} & \mathcal{B} \end{array}$$

is exact.
— *The square above is a contractible square.*
— *The square is fibering and cofibering.*
— *There is a natural transformation $\chi : D_0 \to D_1$ such that $\chi I_{\mathrm{Id}_U} = \mathrm{Id}_\mathcal{B}$.*

5. Absolute exact squares

5.1. Definition of absolute exact squares

Definition 5.1. We define an *absolute exact square* or *absolutely exact square* in Cat as a 2-square $\varphi : US \Rightarrow VT : \mathcal{A}_\mathcal{X} \diamond Y^\mathcal{B}$

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{Y} \\ {\scriptstyle S}\downarrow & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle V} \\ \mathcal{X} & \xrightarrow[U]{} & \mathcal{B} \end{array}$$

such that for any 2-functor $\Phi : \mathrm{Cat} \to \mathrm{Cat}$, the 2-square

$$\Phi(\varphi) : \Phi(U)\Phi(S) \Rightarrow \Phi(V)\Phi(T) : \Phi(\mathcal{A})_{\Phi(\mathcal{X})} \diamond {}^{\Phi(\mathcal{Y})}\Phi(\mathcal{B}) :$$

$$\begin{array}{ccc} \Phi(\mathcal{A}) & \xrightarrow{\Phi(T)} & \Phi(\mathcal{Y}) \\ {\scriptstyle \Phi(S)}\downarrow & \overset{\Phi(\varphi)}{\nearrow} & \downarrow{\scriptstyle \Phi(V)} \\ \Phi(\mathcal{X}) & \xrightarrow[\Phi(U)]{} & \Phi(\mathcal{B}) \end{array}$$

is exact (see Definition 1.13) i.e. such that its virtual right reversion $(\Phi(\varphi))^r$ is invertible.

Remark 5.2. 1 — Of course, as the functors $(-)^{\mathcal{I}}$ are in fact 2-functors from Cat to Cat, an absolute exact square is, in particular, a contractible exact square. So the right challenge is to precise, among the contractible squares, those which are absolute. The difficulty a priori is reduced to the observation in a given contractible square of the part played in the description via Proposition 2.3 by the functors *starting* from $U \downarrow V$.
2 — Because of Proposition 2.3, a contractible exact square is preserved by every 2-functor Φ which preserves comma squares. So the problem is with 2-functors which do not preserve comma squares.
3 — If φ is contractible exact and if the use of $U \downarrow V$ could be eliminated there (in the criterion of Proposition 2.3), and the conditions reduced to equational conditions between functors between the given categories $\mathcal{A}$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{B}$, the given functors S, T, U and V, and some new functors, then the contractible square would be in fact absolute. We will give examples in sections 5.3 and 5.5.
4 — Of course it is clear that:

Proposition 5.3. *The adjunction* 2*-squares* ϵ *and* η *are absolute exact squares, and it is also the case for right or left Beck-Chevalley squares (i.e. exact squares with right adjoints for* T *and for* U*, or exact squares with left adjoints for* S *and* V*).*

5.2. Absolute colimits and absolutely absolute colimits

Paré [23][24][25] introduced the notion of an *absolute coequalizer*, and then more generally the notion of an *absolute colimit* in a category $\mathcal{X}$ as a colimit V in $\mathcal{X}$ given by a co-cone $\varphi : S \Rightarrow V^{\dagger}$ of a diagram $S : \mathcal{A} \to \mathcal{X}$ which is preserved by every functor with domain $\mathcal{X}$, and this is equivalent to the fact that it is preserved by the $\hom_{\mathcal{X}}(X, -) = \mathcal{X}(X, -)$ for every object X in $\mathcal{X}$, and for that it is enough to have the preservation by the functors $\mathcal{X}(S(A), -)$ for every object A in $\mathcal{A}$ and by $\mathcal{X}(V, -)$. Paré obtained the following diagrammatical characterization:

Proposition 5.4. [Paré] *The co-cone* $\varphi : S \Rightarrow V^{\dagger}$ *is an absolute colimit in* $\mathcal{X}$ *if and only if there exist* R *in* $\mathcal{A}$ *and a morphism* $\psi : V \to S(R)$ *in* $\mathcal{X}$ *such that* $\psi.\varphi_R = \mathrm{Id}_V$ *and such that for every* A *in* $\mathcal{A}$ *the two objects* $\mathrm{Id}_{S(A)}$ *and* $\psi.\varphi_A$ *are connected by a zig-zag in* $S(A)^{\dagger} \downarrow S$.

In fact this result was the first step at the root of the proposal of the notion of an exact square in [11] (the link with exact sequences in abelian categories had been observed in a second step, in the same paper); we have:

Proposition 5.5. *The co-cone* $\varphi : S \Rightarrow V^{\dagger}$ *is an absolute colimit in* $\mathcal{X}$ *if and only if the* 2*-square*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathbf{1} \\ {\scriptstyle S}\downarrow & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle V^{\dagger}} \\ \mathcal{X} & \xrightarrow[\mathrm{Id}_{\mathcal{X}}]{} & \mathcal{X} \end{array}$$

is an exact square.

Of course we should not confuse the two uses of "absolute” as in the terms “absolute exact square” and ”absolute colimit”. Rather these two uses could be combined:

Definition 5.6. If the square in Proposition 5.5 is an absolute exact square (according to Definition 5.1) then the co-cone $\varphi : S \Rightarrow V^{\dagger}$ is said to be an *absolutely absolute colimit.*

These absolutely absolute colimits could be characterized diagrammatically (Proposition 5.19) by a special application of Proposition 5.13.

5.3. Absolute and absolutely absolute left Kan extensions

The study of Paré of absolute colimits has been extended to the case of absolute left Kan extension in [16] and after in [10]. So we have:

Proposition 5.7. *The transformation $\varphi : S \Rightarrow VT$ is an* absolute left Kan extension *in $\mathcal{X}$ (i.e. preserved by every functor with domain $\mathcal{X}$) if and only if the 2-square*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{Y} \\ {\scriptstyle S}\downarrow & \overset{\varphi}{\Nearrow} & \downarrow{\scriptstyle V} \\ \mathcal{X} & \xrightarrow[\mathrm{Id}_{\mathcal{X}}]{} & \mathcal{X} \end{array}$$

is an exact square.

For example as a corollary we get:

Proposition 5.8. *A functor $T : \mathcal{A} \to \mathcal{Y}$ is final if and only if the functor $T \downarrow \mathrm{Id}_{\mathcal{Y}} \to \mathcal{Y}$ has connected non-empty fibers.*

Proof. From [29] we know that T is final if and only if $\mathcal{Y} \to \mathbf{1}$ is an absolute left Kan extension of $\mathcal{Y} \to \mathbf{1}$ along T, i.e., with Proposition 5.7 if and only if

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{T} & \mathcal{Y} \\ \downarrow & \overset{=_1}{\Nearrow} & \downarrow \\ \mathbf{1} & \longrightarrow & \mathbf{1} \end{array}$$

is exact; and from Proposition 1.18 this means that the indicator

$$J_{=_1} : T \downarrow \mathrm{Id}_{\mathcal{Y}} \to \mathcal{Y}$$

has connected non-empty fibers. □

Definition 5.9. A functor $P : \mathcal{A} \to \mathcal{B}$ is *opaque* if with $\mathcal{B}_P$ the full subcategory of $\mathcal{B}$ generated by the $P(A)$ with the A in $\mathcal{A}$, the square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{P'} & \mathcal{B}_{\mathbf{P}} \\ {\scriptstyle P}\downarrow & = & \downarrow{\scriptstyle i} \\ \mathcal{B} & \xrightarrow[\mathrm{Id}_{\mathcal{B}}]{} & \mathcal{B} \end{array}$$

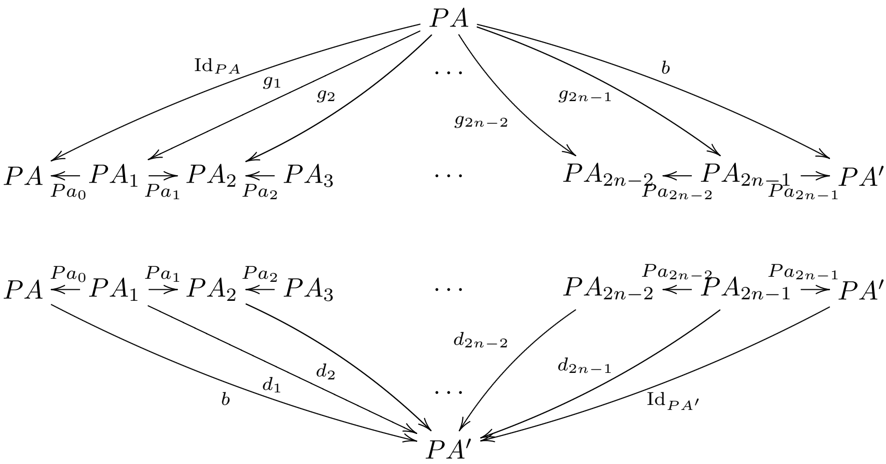
is exact. If furthermore P is surjective on objects, P is said to be *fully opaque.*

In [11] this notion is introduced for the needs of shape theory, and we get the following criterion:

Proposition 5.10. *A functor $P : \mathcal{A} \to \mathcal{B}$ is opaque if and only if for every objects A, A' in $\mathcal{A}$ and every morphism $b : P(A) \to P(A')$ in $\mathcal{B}$ there is in $\mathcal{A}$ a zig-zag*

$$A \xleftarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xleftarrow{a_2} A_3 \ldots\ldots A_{2n-2} \xleftarrow{a_{2n-2}} A_{(2n-1)} \xrightarrow{a_{2n-1}} A'$$

and in $\mathcal{B}$ a lantern

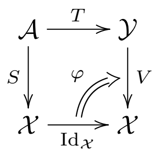


Proposition 5.11. *A fully opaque functor is an epimorphism in* Cat, *but an epimorphism in* Cat *is not necessarily opaque. For functors which are bijective on objects, the two notions coincide.*

Proof. This results from Proposition 5.10 and from the zig-zag theorem of Isbell [17]. □

Theoretically the study of all absolute Kan extensions could be reduced to the study of fully opaque functors, according to the following proposition of Luc Van den Bril quoted in [10], reformulated here with the indicator J_φ:

Proposition 5.12. *The* 2*-square*

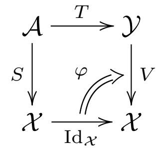


is an exact square if and only if

$$J_\varphi : S\nabla T \to \mathrm{Id}_{\mathcal{X}} \downarrow V$$

is fully opaque.

Proposition 5.13. *For a 2-square $\varphi : US \Rightarrow VT$ where $\mathcal{B} = \mathcal{X}$ and $U = \mathrm{Id}_X$*



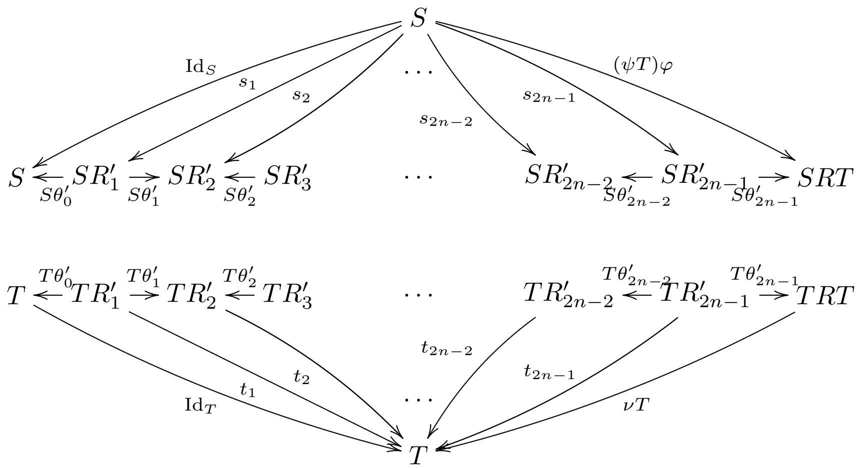
the following conditions are equivalent:
1 — The square is a contractible exact square, i.e. $\varphi^{\mathcal{I}}$ is exact for every category $\mathcal{I}$.
2 — The condition in 1 holds for $\mathcal{I} = \mathcal{Y}$ and for $\mathcal{I} = \mathcal{A}$.
3 — The transformation $\varphi : S \Rightarrow VT$ determines V as a contractible absolute left Kan extension *(or inductive Kan extension), that is to say for every category $\mathcal{I}$ and functor $F : \mathcal{X}^{\mathcal{I}} \to \mathcal{Z}$, $F.\varphi^{\mathcal{I}} : F.S^{\mathcal{I}} \Rightarrow F.V^{\mathcal{I}}T^{\mathcal{I}}$ determines $F.V^{\mathcal{I}}$ as the Kan extension of $F.S^{\mathcal{I}}$ along $T^{\mathcal{I}}$:*

$$\mathrm{Lan}_{T^{\mathcal{I}}}(F.S^{\mathcal{I}}) \simeq F.V^{\mathcal{I}}.$$

4 — There is a functor $R : \mathcal{Y} \to \mathcal{A}$, two natural transformations $\psi : V \to SR$ and $\nu : TR \to \mathrm{Id}_{\mathcal{Y}}$ such that $(V\nu)(\varphi R)\psi = \mathrm{Id}_V$, and a sequence of functors $R'q : \mathcal{A} \to \mathcal{A}$ and in $\mathcal{A}^{\mathcal{A}}$ a zig-zag

$$\mathrm{Id}_{\mathcal{A}} = R'_0 \xleftarrow{\theta'_0} R'_1 \xrightarrow{\theta'_1} R'_2 \ldots R'_{2n-2} \xleftarrow{\theta'_{2n-2}} R'_{2n-1} \xrightarrow{\theta'_{2n-1}} R'_{2n} = R.T,$$

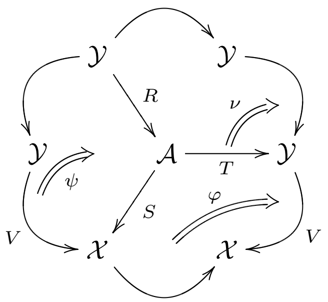
and a lantern



5 — The square is an absolute exact square.
6 — The transformation $\varphi : S \Rightarrow VT$ determines V as an absolutely absolute left Kan extension *(or inductive Kan extension), that is to say that for every 2-functor $\Phi : \mathrm{Cat} \to \mathrm{Cat}$ and functor $F : \Phi(\mathcal{X}) \to \mathcal{Z}$, the natural transformation $F.\Phi(\varphi) : F\Phi(S) \Rightarrow F.\Phi(V)\Phi(T)$ determines $F.\Phi(V)$ as the Kan extension of $F.\Phi(S)$ along $\Phi(T)$:*

$$\mathrm{Lan}_{\Phi(T)}(F.\Phi(S)) \simeq F.\Phi(V).$$

Proof. A proof is possible by application of Proposition 2.3. We let it to the reader: as observed in Remark 5.2 the difficulty is to eliminate the use of $\mathrm{Id}_{\mathcal{X}} \downarrow V$. Another proof works as follows. Clearly, by Definition 2.1and Definition 5.1 and Proposition 5.7, the condition 1 is equivalent to 3, the condition 5 is equivalent to 6. Furthermore the condition of contractible square for $\mathcal{I} = \mathcal{Y}$ implies the existence of $R : \mathcal{Y} \to \mathcal{A}$, ψ and ν arranged in hexagon as in



$$= \quad (V\nu)(\varphi R)(\psi) \quad = \quad \mathrm{Id}_V$$

(corresponding to a functor $\mathcal{A} \to S\nabla T$) and then for $\mathcal{I} = \mathcal{A}$, RT and $\mathrm{Id}_{\mathcal{A}}$ have to be connected over φ. On this other side, the conditions in 4 are preserved by every 2-functor from Cat to Cat, and by composition with every functor $\mathcal{X} \to \mathcal{Z}$, and then it is enough to show that these conditions imply that (V, φ) is an inductive Kan extension of S along T. For that if $W : \mathcal{Y} \to \mathcal{X}$ and $\mu : S \Rightarrow WT$ are given, then with $\tau : V \Rightarrow W$ given by

$$\tau = (W\nu)(\mu R)\psi : V \Rightarrow W$$

we have

$$(\tau T)\varphi = \mu,$$

because of the condition 4 — where the lantern connects $(\psi T)\varphi$ and (νT) to Id_S and Id_T —, and such a τ is unique:

$$\tau = \tau \mathrm{Id}_V = \tau(V\nu)(\varphi R)\psi = (W\nu)(\tau T R)(\varphi R)\psi = (W\nu)(\mu R)\psi.$$

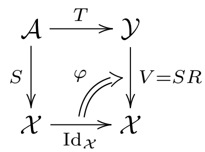
The notations $\sharp$ and $\flat$ used for adjunctions (subsection 1.1.1) could be extended here for the natural bijection between the τ and the μ, with:

$$\mu^\sharp = (W\nu)(\mu R)\psi = \tau, \quad \tau^\flat = (\tau T)\varphi = \mu.$$

□

Remark 5.14. The point in this proof is the reduction of the datum of the functor $K' : \mathrm{Id}_{\mathcal{X}} \downarrow V \to \mathcal{A}$ in Proposition 2.3 and the attendant 2-cells to a datum of $\mathcal{Y} \to \mathcal{A}$ and some attendant 2-cells (elimination of the comma construction, see Remark 5.2-3); conceptually this can be seen as a consequence of the comma object form of Yoneda lemma – see [27] Corollary (16). After that, the system of conditions becomes fully equational in the 2-category Cat, and thus absolute.

Proposition 5.15. *If the functor $T : \mathcal{A} \to \mathcal{Y}$ has a right adjoint R, $T \dashv R(\epsilon, \eta)$, with $\eta : \mathrm{Id}_{\mathcal{A}} \Rightarrow RT$ and $\epsilon : TR \Rightarrow \mathrm{Id}_{\mathcal{Y}}$, then for every functor $S : \mathcal{A} \to \mathcal{X}$, we get an absolute exact square*



where $V = SR$ and $\varphi = S\eta$. So SR is an absolutely absolute left Kan extension of S along T. Especially T is final and even absolutely final.

Proof. A direct proof is easy, but we prefer to do the job as an application of Proposition 5.13. We take $R = R$, $\psi = \mathrm{Id}_{SR}, \nu = \epsilon$. By the adjunction equation we have $(\epsilon_T)(T\eta) = \mathrm{Id}_T$, and as $(S\eta).\mathrm{Id}_S = S\eta$, the lantern condition is satisfied with a zig-zag of one arrow : $\eta : \mathrm{Id}_{\mathcal{A}} \to RT$. And we have also $(V\nu)(\varphi R)\psi = \mathrm{Id}_V$, by the adjunction equation $(R\epsilon)(\eta_R) = \mathrm{Id}_R$.
If T has a right adjoint R, then for any $\mathcal{I}$, $R^{\mathcal{I}}$ is right adjoint to $T^{\mathcal{I}}$ (see Proposition 5.3) , and the first part of this Proposition 5.15 for $T^{\mathcal{I}}$ and $\mathcal{X} = \mathbf{1}$ joined to the argument in the proof of Proposition 5.8 imply that T is absolutely final according to Definition 5.16. □

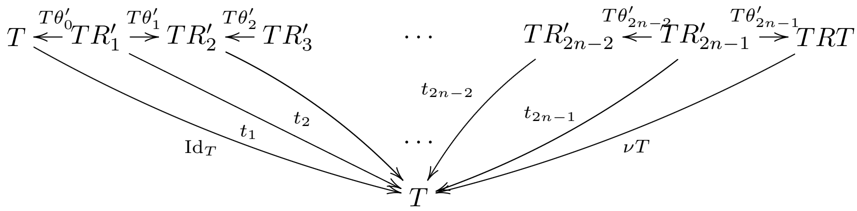
5.4. Criterion for absolutely final functor

Definition 5.16. A functor $T : \mathcal{A} \to \mathcal{Y}$ is *absolutely final* if and only if for any 2-functor $\Phi : \mathrm{Cat} \to \mathrm{Cat}$, the functor $\Phi(T) : \Phi(\mathcal{A}) \to \Phi(\mathcal{Y})$ is final.

Proposition 5.17. *A functor $T : \mathcal{A} \to \mathcal{Y}$ is* absolutely final *if and only if for any category $\mathcal{I}$ the functor $T^{\mathcal{I}} : \mathcal{A}^{\mathcal{I}} \to \mathcal{Y}^{\mathcal{I}}$ is final, if and only there is a functor $R : \mathcal{Y} \to \mathcal{A}$, a natural transformation $\nu : TR \to \mathrm{Id}_{\mathcal{Y}}$, a sequence of functors $R'q : \mathcal{A} \to \mathcal{A}$ and in $\mathcal{A}^{\mathcal{A}}$ a zig-zag*

$$\mathrm{Id}_{\mathcal{A}} = R'_0 \overset{\theta'_0}{\longleftarrow} R'_1 \overset{\theta'_1}{\longrightarrow} R'_2 \ldots R'_{2n-2} \overset{\theta'_{2n-2}}{\longleftarrow} R'_{2n-1} \overset{\theta'_{2n-1}}{\longrightarrow} R'_{2n} = R.T,$$

and a semi-lantern in $\mathcal{Y}^{\mathcal{A}}$



Furthermore this is equivalent to the fact that the two functors

$$T^{\mathcal{A}} \downarrow \mathrm{Id}_{\mathcal{Y}^{\mathcal{A}}} \to \mathcal{Y}^{\mathcal{A}} \quad \text{and} \quad T^{\mathcal{Y}} \downarrow \mathrm{Id}_{\mathcal{Y}^{\mathcal{Y}}} \to \mathcal{Y}^{\mathcal{Y}}$$

have connected non-empty fibers.

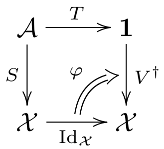
Proof. With $\mathcal{X} = \mathbf{1}$, the first assertion is an application of Proposition 5.13-1 and -4, and exactness of the square in Proposition 4.2 for $F = \Phi(\mathbf{1}) \to \mathbf{1}$. The second one results from Proposition 5.13-2 and Proposition 5.8. □

Remark 5.18. 1 — A left adjoint is absolutely final (Proposition 5.15).
2 —A category $\mathcal{C}$ is contractible (Definition 3.1) if and only if $\mathcal{C} \to \mathbf{1}$ is absolutely final (Proposition 3.7).

5.5. Criterion for absolutely absolute colimits

To conclude we examine the case of absolutely absolute colimits.

Proposition 5.19. *1 — The* 2*-square*



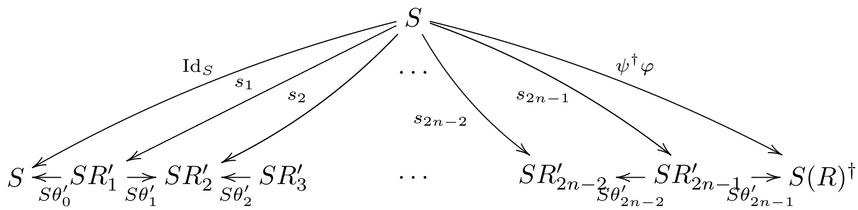
is an absolute exact square — i.e. determines V as an absolutely absolute colimit *of S (Definition* 5.6*) if and only if there is an object R in $\mathcal{A}$ and a morphism $\psi : V \to S(R)$ in $\mathcal{X}$ with*

$$\varphi_R \psi = \mathrm{Id}_V$$

and a contraction of $\mathcal{A}$ on R, i.e. a ziz-zag in $\mathcal{A}^{\mathcal{A}}$

$$\mathrm{Id}_{\mathcal{A}} = R'_0 \xleftarrow{\theta'_0} R'_1 \xrightarrow{\theta'_1} R'_2 \ldots R'_{2n-2} \xleftarrow{\theta'_{2n-2}} R'_{2n-1} \xrightarrow{\theta'_{2n-1}} R'_{2n} = R^\dagger,$$

such that in $\mathcal{X}^{\mathcal{A}}$ the image of this zig-zag by the composition with S provides a connection between Id_S and $\psi^\dagger\varphi$:



In other words: there exist R in $\mathcal{A}$ and a morphism $\psi : V \to S(R)$ in $\mathcal{X}$ such that $\psi.\varphi_R = \mathrm{Id}_V$ and such that the two objects $(\mathrm{Id}_S; \mathrm{Id}_{\mathcal{A}})$ and $(\psi^\dagger.\varphi; R^\dagger)$ are connected by a zig-zag in $S^\dagger \downarrow S^{\mathcal{A}}$, with $S^{\mathcal{A}} : \mathcal{A}^{\mathcal{A}} \to \mathcal{X}^{\mathcal{A}}$ the composition with S, and with $S^\dagger : \mathbf{1} \to \mathcal{X}^{\mathcal{A}}$ the constant functor on S.

Proof. We have only to directly apply Proposition 5.13 to the case $\mathcal{Y} = \mathbf{1}$. □

Remark 5.20. Of course given an absolutely absolute colimit $V \simeq \varinjlim S$, any 2-functor $\Phi : \mathrm{Cat} \to \mathrm{Cat}$ and any object X in $\Phi(\mathbf{1})$ given by $X^\dagger : \mathbf{1} \to \Phi(\mathbf{1})$, we get an isomorphism

$$\Phi(V^\dagger)(X) \simeq \varinjlim \big(\Phi(T) \downarrow X^\dagger \to \Phi(\mathcal{A}) \overset{\Phi(S)}{\to} \Phi(\mathcal{X})\big).$$

It is easy to compare this criterion in Proposition 5.19 with the Paré's criterion for absolute colimits (Proposition 5.4): clearly an absolutely absolute colimit is an absolute colimit. But the converse is false:

Proposition 5.21. *1 — In an absolutely absolute colimit the indexing category $\mathcal{A}$ has to be contractible (in the sense of Definition* 3.1*). As a consequence a coequalizer seen as a colimit on $\mathbf{2}_2$ is never absolutely absolute.*
2 — Splittings of idempotents seen as colimits on $\mathbf{1}^{(2)}$ are absolutely absolute.
3 — The notion of absolutely absolute colimit has examples and is stronger than the notion of absolute colimit.

Proof. 1 — There is a contraction from $\mathrm{Id}_{\mathcal{A}}$ to R, so $\mathcal{A}$ is contractible (Definition 3.1). Then we remark that the indexing category $\mathcal{A}$ for a coequalizer is $\mathbf{2}_2$ which is not contractible (Proposition 3.4), and so stricto sensus there is no absolutely absolute coequalizer.
2 — Let $\mathbf{1}^{(2)}$ be the category with one object R and one non-identity arrow $u : R \to R$ with $u^2 = u$. The final functor $\mathbf{2}_2 \to \mathbf{1}^{(2)}$ determines a splitting of idempotent as a coequalizer. Let (q, j) be the splitting in $\mathcal{X}$ of $a : X \to X$ ($a^2 = a$), with $q : X \to Q$, $j : Q \to X$, $q.j = \mathrm{Id}_Q$, $j.q = a$. We verify conditions in Proposition 5.19 with $\mathcal{A} = \mathbf{1}^{(2)}$, $S(R) = X$ and $S(u) = a$. We have $SR(*)^\dagger = X^\dagger$, $n = 1$, $\theta'_0 = \mathrm{Id}_{\mathrm{Id}_{\mathcal{A}}}$, $(\theta'_1)_R = u$, $S(\theta'_1)_R = a$, $\varphi_R = q$, $\psi_R = j$.
3 — As absolute coequalizers do exist [24], we conclude. □

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