

# Autocategories: I. A Common Setting for Knots and 2-Categories\*

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**Abstract.** An *autograph* is an action of the free monoid with 2 generators; it could be drawn with no use of objects, by arrows drawn between arrows. As examples we get knot diagrams as well as 2-graphs. The notion of an *autocategory* is analogous to the notion of category, by replacing the underlying graph by an autograph. Examples are knots or links diagrams (unstratified case), categories, 2-categories, double categories (stratified case), which so live in the same context, the category of autocategories.

**Résumé.** Un *autographe* est une action du monoïde libre à deux générateurs  $d$  et  $c$ , et peut être représenté en dessinant des flèches entre des flèches, sans utiliser d'objets. Par exemple nous avons les graphes et les 2-graphes. La notion d'*autocatégorie* est semblable à celle de catégorie, en remplaçant le graphe sous-jacent par un autographe. Les exemples sont les diagrammes de nœuds ou d'entrelacs (cas non-stratifiés), les catégories, 2-catégories et catégories doubles (cas stratifiés), qui ainsi résident dans la même catégorie des autocatégories.

**Keywords.** knot, graph, 2-category

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## 1. Autograph

**Definition 1.1.** An autograph  $(A, (d, c))$  is a set  $A$  of elements named arrows, equipped with two maps, domain  $d : A \rightarrow A$  and codomain  $c : A \rightarrow A$ , i.e. a map  $\partial = (d, c) : A \rightarrow A \times A$ . Of course it is the same thing that an action on  $A$  of  $\mathbb{FM}(2) = \{d, c\}^*$ , the free monoid on two generators  $d$  and  $c$ .

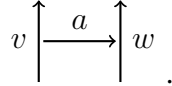
We denote by  $\mathbf{Agraph}$  the category of autographs, with morphism maps  $f : A \rightarrow A'$  with  $d'fa = fda$ ,  $c'fa = fca$ , and  $U : \mathbf{Agraph} \rightarrow \mathbf{Set}$  the forgetful functor given by  $U((A, (d, c))) = A$ .

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**Example 1.2.** For  $(G, (\delta, \gamma))$  a 2-generated group  $G$  (e.g. any finite simple group), with generators  $\delta$  and  $\gamma$ , any  $G$ -set  $E$  is an autograph, with  $d$  and  $c$  given by the actions of  $\delta$  and  $\gamma$  on  $E$ . In such a case  $d$  and  $c$  are invertible.

**Remark 1.3.** We represent the fact that the domain of an arrow  $a \in A$  is the arrow  $v$  and its codomain is the arrow  $w$ , i.e.  $da = v$  and  $ca = w$ , by:  $a : v \rightarrow w$ , or  $v \xrightarrow{a} w$ , or by the picture



**Definition 1.4.** 1 .[path] — A  $(d, c)$ -path or a path of length  $k$  in an autograph  $(A, (d, c))$  is a finite sequence of consecutive arrows  $(z_n)_{0 \leq n \leq k-1}$  with

$$cz_0 = dz_1, \quad cz_1 = dz_2, \quad \dots \quad cz_{k-2} = dz_{k-1}.$$

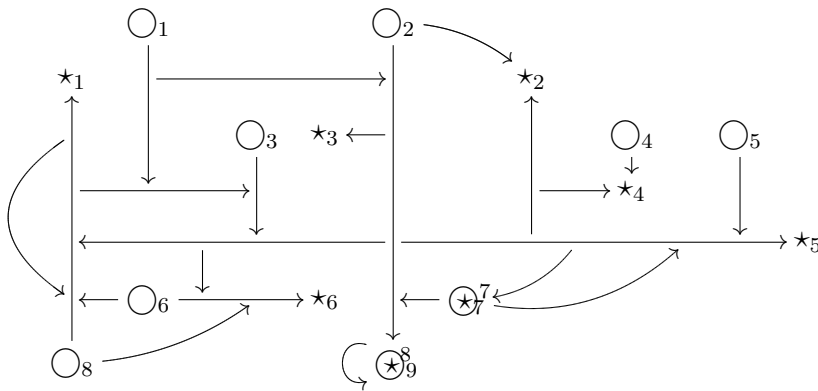
If there is no path of length  $> 1$ , then the autograph is U-free. The set of paths in  $A$  is denoted  $\text{Path}(A, (d, c))$  or shortly  $\text{Path}(A)$ .

2 .[descent] — Given an autograph  $(A, (d, c))$ , a  $(d, c)$ -sequence or a downward sequence or a descent in  $(A, (d, c))$  is a sequence  $(x_n)_{n \geq 0}$  — finite or not — of elements of  $A$  with:

$$\forall n \geq 0 \quad x_{n+1} \in \{dx_n, cx_n\}.$$

If there is no cyclic (resp. infinite) descent, then the autograph is stratifiable (resp. foundable). The set of  $(d, c)$ -sequences or descents in  $A$  is denoted  $\text{Desc}(A, (d, c))$  or shortly  $\text{Desc}(A)$ .

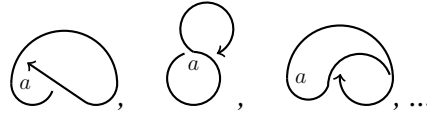
**Example 1.5.** A random example of a fragment of an autograph is:



where the  $\bigcirc_j$  are sources, and the  $\star_i$  are targets, with  $\bigcirc_9 = \star_8$ , in such a way that the picture looks like a kind of super-arrow from  $(\bigcirc_j)_{j=1,\dots,9}$  toward  $(\star_i)_{i=1,\dots,8}$ ; the sources and targets are ‘open’ or ‘empty’ places, in the sense that they must be filled by new arrows, for example by auto-arrows (cf. ?? and ??).

## 2. Auto-arrows and terminal autograph, self-reference

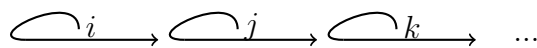
**Definition 2.1.** In an autograph, an ‘auto’-arrow is an auto-mapping from itself to itself, i.e. an arrow  $a$  with domain  $da = a$  and codomain  $ca = a$ , i.e. a data which could be written as “ $a : a \rightarrow a$ ”, and be drawn as



**Remark 2.2.** Of course very often — by way of *abbreviation* — we denote an ‘auto’-arrow by a simple closed curve, as a circle  $\bigcirc_a$ , or a square  $\square_a$ , or even by a bullet  $\bullet_a$ , or a star  $\star_a$ , or a crossing  $\times_a$ , etc., and so we get a ‘point’ in our picture; but such a ‘point’ is not at all considered as a static or stable ‘object’, rather it is an auto-modification.

**Definition 2.3.** The terminal autograph, i.e. the terminal object in *Agraph*, consists of one letter named  $*$ ; it will be denoted by  $S_* = \{ * : * \rightarrow * \}$  (referencing to the shape ‘S’ of the second picture for an auto-arrow in ??). An auto-arrow  $a$  in  $A$  is equivalent to a morphism of autographs  $a^\bullet : S_* \rightarrow (A, (d, c))$ , the constant map on  $a$ .

**Remark 2.4.** The visualization of *self-reference* as an auto-arrow, or a ‘partial auto’-arrow was introduced by Jean Schneider [?, ?], in order to modelize graphically the structure of time : an instant  $i$  is an operation  $i$  applied to itself  $i$  and producing a new thing  $j$ , which itself produces  $k$ , etc. and so the time is constructed:



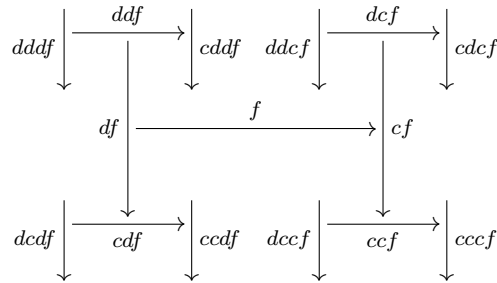
### 3. Free autographs on $\aleph_0$ generators

**Proposition 3.1.** *The U-free autograph on  $\aleph_0$  generators  $\text{FA}(3\mathbb{N})$  can be constructed as  $(\mathbb{N}, (t_1, t_2))$ , with  $\mathbb{N}$  the set of natural numbers and  $d = t_1$ ,  $c = t_2$ :*

$$t_1(m) = 3m + 1, \quad t_2(m) = 3m + 2 \quad (\star)$$

*Any finite or denumerable example of autograph is a quotient of this one. Furthermore, the set  $\mathbb{R}$  of real numbers is identified to a subset of the set of descents in  $\text{FA}(3\mathbb{N})$ :  $\mathbb{R} \subset \text{Desc}((\mathbb{N}, (t_1, t_2)))$ .*

*Proof.* 1 — The free autograph  $\text{FA}(\{f\})$  on one generator  $f$  starts with:



According to a dyadic process, the beginning of  $\text{FA}(\{f\})$  is pictured as this given H-binary tree. It could also be seen as a part of the Cayley graph for the free group on two generators  $s, t$  [?, Fig. 2.3., p.40] with  $f \mapsto 1$ ,  $df \mapsto s^{-1}$ ,  $cf \mapsto s$ ,  $ddf \mapsto ts^{-1}$ ,  $cdf \mapsto t^{-1}s^{-1}$ ,  $dcf \mapsto ts$ ,  $ccf \mapsto t^{-1}s$ , etc.

2 — With  $d = 1$  and  $c = 2$ , and with  $f = .$ , to each element of  $\text{FA}\{f\}$  is associated a triadic code with no 0 ; for example to  $dcccdf$  is associated the code .212221, and the associated rational number  $\frac{2}{3} + \frac{1}{9} + \dots + \frac{1}{729} = \frac{403}{729}$ . At this level, the operations  $d$  and  $c$  are realized as  $d = T_1$  and  $c = T_2$ :

$$T_1\left(\frac{m}{n}\right) = \frac{m}{n} + \frac{1}{3} \frac{1}{n}, \quad T_2\left(\frac{m}{n}\right) = \frac{m}{n} + \frac{1}{3} \frac{2}{n}. \quad (\star\star)$$

Then  $\text{FA}(\{f\})$  appears as a sub-autograph of the one consisting in the set  $[0, 1]_{\text{rat}} = \left\{\frac{m}{n}; 0 \leq m < n\right\}$  equipped with  $d$  and  $c$  given by  $(\star\star)$ .

In fact, 403 determines completely the fraction  $\frac{403}{729}$ , the sequence 12221 being obtainable by successive divisions by 3, as the successive residues. So  $\text{FA}\{f\}$  appears also as a sub-autograph of the one consisting in the set  $\mathbb{N}$



$w = \omega(a)$ , if the first crossing is a crossing of  $v$  and  $a$ , and the second one is a crossing of  $w$  and  $a$ ; so, on the set  $\text{Arc}(\mathbb{K})$  of arcs with these two maps  $d = \alpha$  and  $c = \omega$  we get a structure of autograph, denoted by  $\text{As}(\mathbb{K})$ , in which for each  $a$  we have:

$$\begin{array}{c} a^- \longrightarrow \alpha(a) \quad \uparrow \quad a \longrightarrow \uparrow \quad \omega(a) \longrightarrow a^+ \\ \hline \end{array}$$

□

**Remark 4.2.** Elements of  $\text{Arc}(\mathbb{K})$  are not completely ‘abstract’, they are real arcs in the plan, and so we can precise a sign for each crossing :

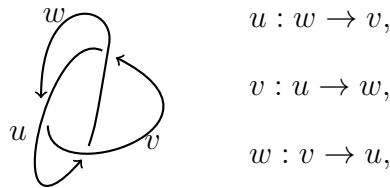
$$(+1) \quad \begin{array}{c} w \swarrow \quad \nearrow a^+ \\ \nearrow a \quad \swarrow \end{array} \quad (-1) \quad \begin{array}{c} w \swarrow \quad \nearrow a^+ \\ \swarrow a \quad \nearrow \end{array}$$

So each arc  $a$  of the oriented knot diagram is equipped with a double sign  $(\epsilon, \eta)$ , where  $\epsilon$  is the sign of its initial crossing and  $\eta$  is the sign of its final crossing; the data  $(a, (\epsilon, \eta))$  is named a *doubly signed arc*,  $(a, \epsilon\eta = \sigma)$  is named a *signed arc*, and  $a$  itself is an *unsigned arc* (unsigned, but oriented), with a predecessor  $a^-$  (crossing with  $v$ , sign  $\epsilon$ ) and a successor  $a^+$  (crossing with  $w$ , sign  $\eta$ ). Of course the values of  $\epsilon$  and  $\eta$  determine the orientations on  $d$  and  $c$ .

Now, as the set of unsigned arcs, the set of signed arcs and the set of doubly signed arcs — in the given knot diagram  $\mathbb{K}$  — are two other autographs associated to  $\mathbb{K}$ , denoted by  $\text{As}_\sigma(\mathbb{K})$  and  $\text{As}_{\epsilon,\eta}(\mathbb{K})$ .

Of course the same constructions work for any oriented link diagram  $\mathbb{L}$ .

**Example 4.3.** The simplest example is the *trefoil knot*, with an oriented diagram  $\mathbb{T}$ , in which the double sign of each arc is  $(+, +)$ ; the associated autograph  $\text{As}(\mathbb{T})$  is pictured and listed as follows:



or described by :  $\alpha u = v, \omega u = w, \alpha v = w, \omega v = u, \alpha w = u, \omega w = v$ .

**Remark 4.4.** In [?, p. 168-169], this diagram is commented almost as follows. With  $R = {}^v_w$ ,  $S = {}^w_u$ ,  $I = {}^u_v$ , we write, as for fractions (or matrices composition or tensors calculus),  $S \star I = {}^w_u \star {}^u_v = {}^w_v = R^{-1}$ ; hence Schneider think of the trefoil as representing relations like :

$$S \star I = R^{-1}, \quad I \star R = S^{-1}, \quad R \star S = I^{-1},$$

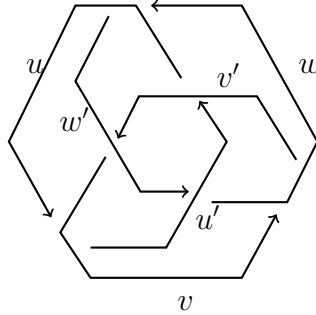
and the trefoil becomes a symbol of a borromean situation between  $R$ ,  $S$  and  $I$ . Then the borromean schema is presented as an enrichment of the trefoil with three new crossing points  $r$ ,  $s$  and  $i$ , with

$$R = I \star i, S = R \star r, I = S \star s, r = i \star I, s = r \star R, i = s \star S.$$

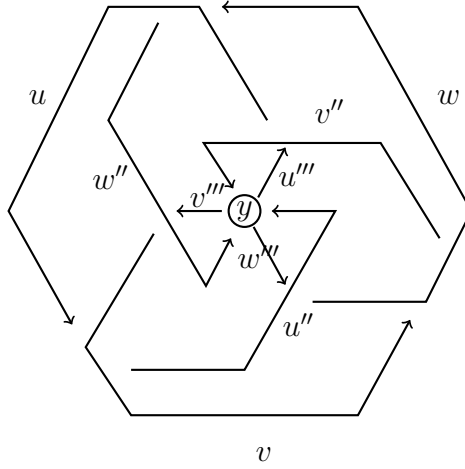
Here we proceed in a different way, trying to stay always at the level of autographs (arrows), i.e. with arcs rather than with crossing points (objects).

**Example 4.5.** An example is the *borromean* link, with an oriented diagram  $\mathbb{B}$ , and the associated autograph  $\text{As}(\mathbb{B})$  is listed and pictured as follows:

$$\begin{aligned} u : v' \rightarrow v, & & u' : v \rightarrow v', \\ v : w' \rightarrow w, & & v' : w \rightarrow w', \\ w : u' \rightarrow u, & & w' : u \rightarrow u'. \end{aligned}$$



**Example 4.6.** Starting with  $\mathbb{B}$ , in its ‘hexagonal center’ we can do a surgical procedure first consisting in the introduction of a Y-cut, cutting  $u'$  near the end in a point  $\alpha$ , now separated into  $\alpha^-$  and  $\alpha^+$ , and similarly for  $v'$  and  $w'$ , and then continued by the junctions of  $\alpha^+$  and  $\beta^-$ ,  $\beta^+$  and  $\gamma^-$ ,  $\gamma^+$  and  $\alpha^-$ . Finally, after three Reidemeister moves of type I (twist)[?] to eliminate the three new loops, we get the picture of the trefoil  $\mathbb{T}$ . The data of this construction determine an autograph  $\text{As}(\mathbb{B})_Y$ , pictured and listed as:



$$\begin{aligned}
 u : v'' \rightarrow v, & \quad v : w'' \rightarrow w, & \quad w : u'' \rightarrow u, \\
 u'' : v \rightarrow y, & \quad v'' : w \rightarrow y, & \quad c'' : a \rightarrow y, & \quad y : y \rightarrow y. \\
 u''' : y \rightarrow v'', & \quad v''' : y \rightarrow w'', & \quad w''' : y \rightarrow u'',
 \end{aligned}$$

In  $\text{As}(\mathbb{B})_Y$ , the Y-cut is simulated by the introduction of an auto-arrow  $y$  and three arrows making the convenient junction:  $u'''$ ,  $v'''$ ,  $w'''$ .

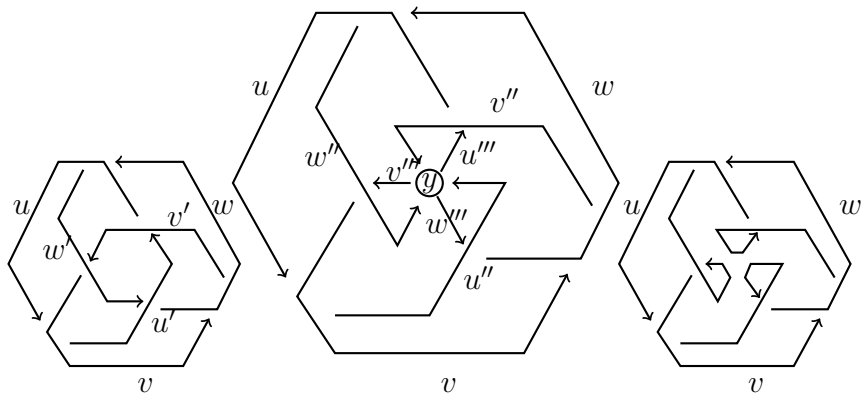
**Proposition 4.7.** *At the level of autographs, the surgery procedure in ?? — explaining how to get  $\mathbb{T}$  from  $\mathbb{B}$  — could be translated by the construction of two maps between three free autocategories of paths on an autograph (see definition ??, ??):*

$$\text{Path}(\text{As}(\mathbb{B})) \xrightarrow{\bar{\beta}} \text{Path}(\text{As}(\mathbb{B})_Y) \xleftarrow{\bar{\tau}} \text{Path}(\text{As}(\mathbb{T})).$$

*Proof.* The reader is invited to follow the paths on the picture of  $A(\mathbb{B})_Y$ , in order to understand the transformations. The left mapping is determined by the map  $\beta : \mathbb{B} \rightarrow \text{Path}(\text{As}(\mathbb{B})_Y)$ , and the right one is determined by  $\tau : \mathbb{T} \rightarrow \text{Path}(\text{As}(\mathbb{B})_Y)$ , with

$$\begin{aligned}
 \beta(u) &= u, \beta(v) = v, \beta(w) = w, \\
 \beta(u') &= u'''u'', \beta(v') = v'''v'', \beta(w') = w'''w'', \\
 \tau(u) &= uu'''v'', \tau(v) = vv'''w'', \tau(w) = ww'''u''.
 \end{aligned}$$



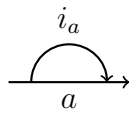


□

## 5. Identifiers, autographs and flexigraphs with identifiers

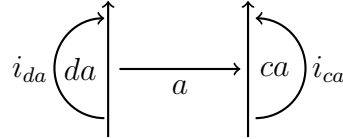
In the context of graphs, diagrams and categories, the notions of vertex, object, identity and unit, are almost identified. With the notion of autograph we clarify a distinction between them, at first by the elimination of the notion of object, and the introduction of the notion of auto-arrow. Secondly now we precise the notion of an *identifier*. In the next section we will introduce units and identities.

**Remark 5.1.** In presence of an object or a vertex  $X$ , in a category or a graph for example, we have to be careful and not to confuse the identity mapping  $1_X : X \rightarrow X$  of the object  $X$  with an auto-arrow on  $X$ ; in such a situation, it is  $X$  itself which determines an auto-arrow  $X : X \rightarrow X$ , or better we have to consider that  $1_X$  is an auto-arrow :  $1_X : 1_X \rightarrow 1_X$  (see ??). But, in the general situation an identity mapping could exist on an arrow different from any object, even for an arrow which is not an auto-arrow, i.e. for such an arrow  $\xrightarrow{a}$ . In fact we could have a data  $i_a : a \rightarrow a$  representing a so called 'identifier' on  $a$ , i.e. a selected endo-arrow of  $a$ ,  $i_a$ , such that  $di_a = a = ci_a$ :



Such a data  $i_a$  determines  $a$  (by  $di_a = a$ ), and so is really an 'identifier' of  $a$  (a 'name' of  $a$ ), but is distinct from  $a$ , and is not an auto-arrow, and is not necessarily unique. Later we will see if a given identifier has to be an identity.

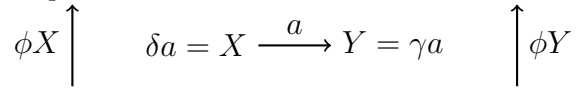
**Definition 5.2.** An autograph  $A$  in which each arrow  $a$  is equipped with the data of a selected endo-arrow  $i_{da} : da \rightarrow da$  named the identifier of  $da$ , and a selected endo-arrow  $i_{ca} : ca \rightarrow ca$  named the identifier of  $ca$ , is called an autograph with identifiers. Then, with each arrow  $a$ , we have the picture



**Remark 5.3.** Clearly  $i_{da}$  (or  $i_{ca}$ ) does not depend really of  $a$ , but only of  $v = da$  (or  $w = ca$ ). Of course in an autograph with identifiers, if  $a$  is the domain (or the codomain) of something, i.e.  $a = db$  or  $a = cb$ , then we have an identifier  $i_a$  on  $a$ . And if  $i_a$  itself is the domain (or codomain) of something, then we have an identifier  $i_{i_a}$  (or  $j_{i_a}$ ), etc. But if  $a$  is not a domain or a codomain, then no identity on  $a$  is assumed to be specified.

**Definition 5.4.** A flexigraph is the data of two sets  $G_0$  and  $G_1$ , and three maps  $\delta, \gamma : G_1 \rightarrow G_0$ ,  $\phi : G_0 \rightarrow G_1$ .

The map  $\phi$  is the flex, and, as in the case of an autograph for  $d$  and  $c$ ,  $\delta$  and  $\gamma$  are thought as ‘domain’ and ‘codomain’. The difference with an autograph is that there are two types of elements [0 (vertex), 1 (arrow)], and then domain and codomain look like ‘objects’ (or ‘vertices’). For any  $a \in G_1$  we get the picture :



There are also ‘isolated’  $\phi Z$ , for any  $Z$  which is not a  $\delta a$  or a  $\gamma a$ .

**Example 5.5.** An autograph  $(A, d, c)$  is a special case of flexigraph, with  $G_1 = A = G_0$ ,  $\delta = d$ ,  $\gamma = c$ , and  $\phi = 1_A$ . Conversely a flexigraph  $(G_1, G_0, \delta, \gamma, \phi)$  determines an autograph with  $A = G_1$ ,  $d = \phi\delta$ ,  $c = \phi\gamma$ . If the flexigraph is a flexigraph with ‘identities’, i.e. is equipped with a map  $\iota : G_0 \rightarrow G_1$  such that  $\delta\iota = 1_{G_0} = \gamma\iota$  — that is to say that  $(G_1, G_0, \delta, \gamma, \iota)$  is an oriented graph (usually named today a graph), and if  $\phi$  is injective — then the associated autograph is with identifiers, with  $i_{da} = \iota\delta a$ ,  $i_{ca} = \iota\gamma a$ . Of course this works in the special case of an oriented graph just seen as a flexigraph with  $\phi := \iota$ .

## 6. Autocategories

**Definition 6.1.** An autocategory  $\mathcal{A}$  is the data of an autograph  $(A, d, c)$  with identifiers (definitions ?? and ??), equipped with the data of a composition law for consecutive arrows, i.e. for any consecutive arrows

$$f : p \rightarrow q, \quad g : q \rightarrow r \quad (\text{where } q = dg = cf),$$

we have a composed arrow denoted  $gf$ , with  $d(gf) = df = p$  and  $c(gf) = cg = r$ , i.e.

$$\text{Position: } gf : p \rightarrow r,$$

such that identities are units for composition:

$$\text{Unitarity: } fi_{df} = f = i_{cf}f;$$

and such that the two compositions of three consecutive arrows are equal:

$$\text{Associativity: } h(gf) = (hg)f, \quad \text{if } dh = cg \quad \text{and} \quad dg = cf.$$

We denote by  $\text{Acat}$  the category of autocategories, with morphism the maps  $F : A \rightarrow A'$  with  $d'Fa = Fda$ ,  $c'Fa = Fca$ , and  $F(ba) = F(b)F(a)$  if  $db = ca$ .

The forgetful functor  $\mathbf{V} : \text{Acat} \rightarrow \text{Agraph}$  is given by  $\mathbf{V}(\mathcal{A}) = (A, d, c)$ .

**Proposition 6.2.** A category determines an autocategory  $\text{Ass}(\mathcal{C})$ . Furthermore, any structure of flexigraph on its underlying graph, such that the flex  $\phi$  is injective, determines another structure of autocategory  $\text{Ass}(\mathcal{C}, \phi)$ , with the “same” arrows and the same composition law (but a very different underlying autograph); such a structure is named a flexicategory or a category with a flexion.

*Proof.* This definition is almost the same as the definition of a category [?], [?], excepted that now there are no objects. As in a category, the identifier  $i_a$  on  $a$ , if it exists, is unique, because it has to be a unit: in this case it is named an *identity*. This identity does exist when  $a$  is a domain or a codomain, but not only in the case where  $a$  is the auto-arrow associated to an object.

So, starting with a category  $\mathcal{C}$ , we get an autocategory  $\text{Ass}(\mathcal{C})$  by replacing each object  $X \in \text{Obj}(\mathcal{C})$  by an auto-arrow  $i_X : i_X \rightarrow i_X$ , where  $i_X$  ‘is’

the arrow  $1_X : X \rightarrow X$ , identity on  $X$  in the category. If in  $\mathcal{C}$  we have  $f : X \rightarrow Y$ , i.e.  $X = \text{dom}(f)$  and  $X = \text{cod}(f)$ , then we consider that  $i_X = d_{\text{Ass}(\mathcal{C})}f$ ,  $i_X = c_{\text{Ass}(\mathcal{C})}(f)$ . Composition is the same as in  $\mathcal{C}$ . Of course in this autcategory we recover the objects as being the auto-arrows which are units (as already it works in categories [?]).

If on  $\mathcal{C}$  we have a flex  $\phi : \text{Obj}(\mathcal{C}) \rightarrow \text{Arrow}(\mathcal{C})$  (see definition ??), we define  $d_\phi$  and  $c_\phi$  by  $d_\phi(f) = \phi(d_{\mathcal{C}}(f))$  and  $c_\phi(f) = \phi(c_{\mathcal{C}}(f))$ , then for every object  $X$ ,  $i_X$  become an arrow from  $\phi(X)$  to  $\phi(X)$ , which is the identity on  $\phi(X)$ .  $\square$

**Proposition 6.3.** *To any autograph  $(A, (d, c))$  there is associated a  $\mathbb{V}$ -free autcategory  $\mathbb{P}(A, (d, c)) = (\text{Path}^t(A, (d, c)), D, C)$ , which is the free autcategory on  $A$ .*

*Proof.* We consider paths  $(z_n)_{0 \leq n \leq k-1}$  in  $A$ , shortly denoted by  $(z_n)_k$  (definition ??), with  $D((z_n)_k) := (dz_0)_0$  and  $C((z_n)_k) = (cz_k)_0$ , and so these paths are between paths of length 1 consisting in a domain or a codomain in  $A$ , i.e. an  $a$  of the form  $dx$  or  $cy$ ; for any of these  $a$  we have to add an identity element  $I_a$  to  $\text{Path}(A)$ . So we get the set  $\text{Path}^t(A)$ . If  $a$  is an identifier  $i_b$  in  $A$ , and possibly an identity when  $A$  is an autcategory, then we should not confuse  $i_b$ ,  $(i_b) = (i_b)_0$ , and  $I_{i_b}$ . In fact  $I_a$  plays the part of the empty sequence in the usual calculus of words; but here we need several empty words, one by domain or codomain  $a$ . Then the composition is given by concatenation of paths, and by the equations  $(z_n)_k I_{z_0} = (z_n)_K$  and  $I_{z_k} (z_n)_k = (z_n)_k$ .  $\square$

## 7. Double categories and 2-categories as autcategories

**Proposition 7.1.** *Any double category  $\mathbb{C}$  (and especially any 2-category) is determined by an associated autcategory  $\text{Ass}(\mathbb{C})$ .*

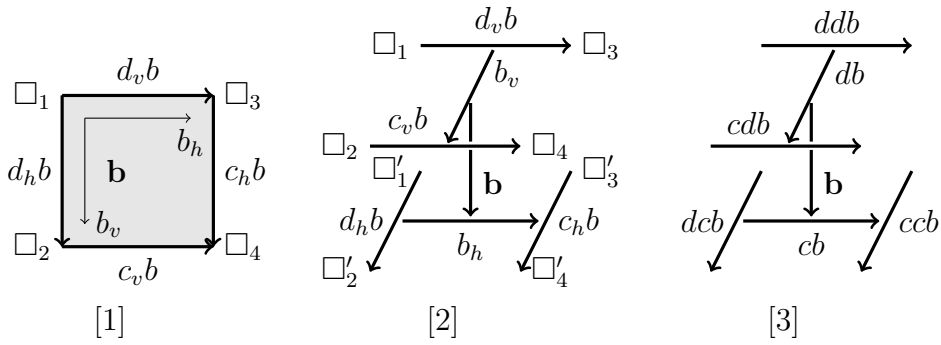
*Proof.* 1 — Let  $\mathbb{C}$  be a double category [?], where  $d_h, d_v, c_h, c_v$  are horizontal and vertical domains and codomains, where  $\infty$  and  $\delta$  are horizontal and vertical compositions. Let  $\underline{\mathbb{C}}$  be the set of all elements in  $\mathbb{C}$  (2-block, horizontal arrow, vertical arrow, or object).

The underlying set of  $\text{Ass}(\mathbb{C})$  will be  $\underline{\mathbb{C}}_v + \underline{\mathbb{C}}_u + \underline{\mathbb{C}}_h$ , a sum of three copies of  $\underline{\mathbb{C}}$ , and an arbitrary element of  $\mathbb{C}$  will have three avatars: if  $x$  is a designation

of any of these avatars, we get designations for all the avatars:  $x_u$  is the un-oriented version,  $x_h$  is the horizontally oriented version,  $x_v$  is the vertically oriented version. So  $(x_h)_h = x_h$ ,  $(x_h)_v = x_v$ ,  $(x_h)_u = x_u$ , etc. Each avatar determines the others. In  $\text{Ass}(\mathbb{C})$  we consider that  $x_u$  is an arrow from  $x_v$  to  $x_h$ , i.e.  $x_u : x_v \rightarrow x_h$ , and so we define:

$$d(x_u) = x_v, \quad c(x_u) = x_h.$$

Let  $\mathbf{b}$  be a not degenerated element of the double category, an unoriented 2-block, a 2-dim data as in picture [1]:



To get a determined operation (of type  $\infty$  or 8) with  $\mathbf{b} = b_u$  we need an orientation: to operate horizontally we use of  $b_h$ , and to operate vertically we use of  $b_v$ . We introduce, successively

$$d(b) = b_v, \quad c(b) = b_h$$

$$ddb = d(b_v) = (d_v b)_h, \quad cdb = c(b_v) = ((c_v b))_h,$$

$$dcb = d(b_h) = (d_h b)_v, \quad cdb = c(b_h) = (c_h b)_v,$$

$$\square_1 = dddb = (d_h d_v b)_v, \quad \square'_1 = ddcb = (d_v d_h b)_h.$$

We have

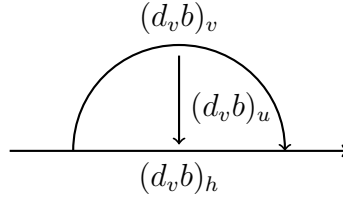
$$(\square_1)_h = \square'_1, \quad (\square'_1)_v = \square_1,$$

And the same facts for the three other corners.

Let us remark that, for clarity and simplicity, in picture [2] and [3] not all the existing arrows are drawn ; for example in fact  $d_v b$  is for  $(d_v b)_h$ , but there are also  $(d_v b)_v$ , which has to be the vertical unity on  $(d_v b)_h$  in  $\mathbb{C}$ , and here in  $\text{Ass}(\mathbb{C})$  its identity

$$i_{(d_v b)_h} = (d_v b)_v,$$

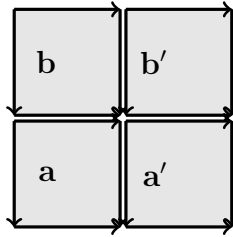
and then we have also  $(d_v b)_u : (d_v b)_v \rightarrow (d_v b)_h$ , etc.



In  $\text{Ass}(\mathbb{C})$ , if  $dy = cx$ , then  $x$  and  $y$  are both horizontal or both vertical. So a unique composition law “.” in  $\text{Ass}(\mathbb{C})$  is defined as follows, if and only if  $dy = cx$ :

$$y.x = \begin{cases} y \infty x & \text{if } y \text{ and } x \text{ are horizontal,} \\ y \delta x & \text{if } y \text{ and } x \text{ are vertical.} \end{cases}$$

Now, to conclude, we have to consider the axiom which relates the two composition laws  $\infty$  and  $\delta$  in  $\mathbb{C}$ , for a ‘square of squares’:



and to translate it in  $\text{Ass}(\mathbb{C})$ .

This compatibility (distributivity) in  $\mathbb{C}$  is

$$(a' \infty a) \delta (b' \infty b) = (a' \delta b') \infty (a \delta b),$$

when

$$d_v a = c_v b, d_v a' = c_v b', d_h a' = c_h a, d_h b' = c_h b.$$

The translation in  $\text{Ass}(\mathbb{C})$  is:

$$[(a'_h \cdot a_h)_v \cdot (b'_h \cdot b_h)_v]_h = (a'_v \cdot b'_v)_h \cdot (a_v \cdot b_v)_h,$$

as well as

$$(a'_h \cdot a_h)_v \cdot (b'_h \cdot b_h)_v = [(a'_v \cdot b'_v)_h \cdot (a_v \cdot b_v)_h]_v.$$

If we introduce on  $\text{Ass}(\mathbb{C})$  the involutive *transversal map*  $(-)^{\theta}$  by

$$(x_h)^{\theta} = x_v, \quad (x_u)^{\theta} = x_u, \quad (x_v)^{\theta} = x_h,$$

such that

$$(x^\theta)^\theta = x, \quad (\text{Inv})$$

then we could recover the identities  $i_d$  and  $i_c$  on domains and codomains:  $i_{dx} = (dx)^\theta$ ,  $i_{cx} = (cx)^\theta$ , in such a way that for any  $x$  we have

$$(cx)^\theta x = x = x(dx)^\theta. \quad (\text{Id})$$

And with  $\theta$  the compatibility becomes:

$$(a'.a)^\theta.(b'.b)^\theta = [(a'.b')^\theta.(a.b)^\theta]^\theta. \quad (\text{Comp})$$

□

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