Autocategories: II. Autographic Algebras*

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Abstract. An *autograph* is a set A with an action of the free monoid with 2 generators; it could be drawn as arrows between arrows. In [5] we have shown that knot diagrams as well as 2-graphs are examples. Of course the category of autographs is a topos, and an *autographic algebra* will be the algebra of a monad on this topos. In this paper we compare autographic algebras with *graphic algebras* of Burroni, via *graphic monoïds* of Lawvere. For that we use monadicity criterions of Lair and of Coppey. The point is that when it is possible to replace graphic algebras by autographic algebras, we change a situation with 2 types of arities into a situation with only 1 type, the type "object" being avoided. So graphs, basic graphic algebras, autographs in a category of algebras of a Lawvere theory, elements of any 2-generated graphic topos, categories, autocategories, associative autographs, are autographic algebras.
Résumé. Un *autographe* est un ensemble A équipé d'une action du monoïde libre à deux générateurs, et peut être représenté en dessinant des flèches entre des flèches. Dans [5] nous avons obtenu comme exemples les diagrammes de nœuds et les 2-graphes. Evidemment la catégorie de ces autographes est un topos, et une *algèbre autographique* sera une algèbre d'une monade sur ce topos. Ici nous comparons ces algèbres avec les algèbres graphiques de Burroni, via les monoïdes graphiques de Lawvere, en utilisant les critères de monadicité de Lair et de Coppey. Le point est que lorsque l'on remplace une situation graphique par une situation autographique, on transforme une situation à 2 types d'arités en une situation à 1 type, le type "objet" étant montré évitable. Ainsi les graphes, les algèbres graphiques basiques, les autographes dans une catégorie d'algèbres de Lawvere, les éléments de topos graphiques 2-engendrés, les catégories, les autocatégories, et les autographes associatifs sont des algèbres autographiques.

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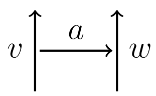
1. From graphs to autographs

Definition 1.1. *(see [5, def. 1.1., p.66]) We denote by* $\mathbb{FM}(2) = \{d, c\}^*$ *the free monoïd on two generators* d *and* c*. As a category with one object* υ*, this monoïd* $\mathbb{FM}(2)$ *is the category of paths in the graph*

Especially the identity is the empty path "()", also denoted by 1_υ*.*
An autograph $(A, (d_A, c_A))$ *is a set* A *of* arrows*, equipped with two maps* domain $d_A : A \to A$ *and* codomain $c_A : A \to A$*; that is to say an action of* $\mathbb{FM}(2)$ *on* A*; if necessary this action is again denoted by* A*, with*

$$A(\upsilon) = A, A(d) = d_A, A(c) = c_A.$$

We represent $a \in A$ with $d_A a = v$ and $c_A a = w$, by: $a : v \to w$, or $v \xrightarrow{a} w$, or by:



The category of autographs is Agraph $= \mathrm{Set}^{\mathbb{FM}(2)}$; in this category a morphism is a map $f : A \to A'$ satisfying $d' fa = fda, \;\; c' fa = fca$. We have a forgetful functor:

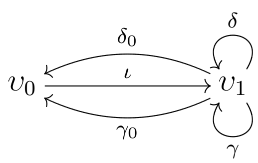
$$\mathrm{U} : \mathrm{Agraph} \to \mathrm{Set} : (A, (d_A, c_A)) \mapsto A.$$

Definition 1.2. *We denote by* $\mathbb{G}(2)$ *the category with two objects* υ_0 *and* υ_1*, and five non-identity arrows*

$$\gamma_0, \delta_0 : \upsilon_1 \to \upsilon_0, \;\; \iota : \upsilon_0 \to \upsilon_1, \;\; \delta, \gamma : \upsilon_1 \to \upsilon_1,$$

with identities on υ_1 *and* υ_0*, and with equations :*

$$\delta_0.\iota = 1_{\upsilon_0}, \;\; \gamma_0.\iota = 1_{\upsilon_0}, \;\; \gamma = \iota.\gamma_0, \;\; \delta = \iota.\delta_0.$$



Presheaves G on $\mathbb{G}(2)$, i.e. objects of Graph $=$ $\mathrm{Set}^{\mathbb{G}(2)}$ are named *graphs*. Any $V \in G(\upsilon_0)$ is named a *vertex*, and if $f \in G(\upsilon_1)$, f is named an *arrow*; then the fact that $G(\delta_0)(f) = V$ and $G(\gamma_0)(f) = V'$ is represented by: $f : V \to V'$.

Remark 1.3. When we work "over graphs" we have to consider 2 types of arities (vertices and arrows), wheras working "over autographs" introduces only 1 type (arrows). So our question here is to understand precisely when the reduction of a 2 types situation to a 1 type situation is possible.

Proposition 1.4. *The comparison between graphs and autographs is induced by pre-composition with the functor*

$$\mathbb{FM}(2) \xrightarrow{\phi} \mathbb{G}(2)$$

given by

$$\phi(\upsilon) = \upsilon_1,\ \phi(d) = \delta,\ \phi(c) = \gamma.$$

Up to an isomorphism, any graph $G : \mathbb{G}(2) \to \mathrm{Set}$ *is determined by its associated autograph* $G\phi : \mathbb{FM}(2) \to \mathrm{Set}$.

Proof. In a graph G for each vertex $V \in G(\upsilon_0)$, the arrow $G(\iota)(V) \in G(\upsilon_1)$ is exactly a fixed point of δ, i.e. an $x \in G(\upsilon_1)$ such that $\delta(x) = x$, as well as exactly a fixed point of γ. So in fact we recover the set of vertices of G as the splitting of the idempotent δ, or also the splitting of γ (these two splittings are isomorphic). □

Proposition 1.5. *The comparison* ϕ *between graphs and autographs in Proposition 1.4 admits a factorisation through the image* $\mathbb{M}(2)$ *of* ϕ*, in such a way that* $\mathbb{G}(2)$ *is the strict karoubian envelope of* $\mathbb{M}(2)$.

$$\mathbb{G}(2) = \mathrm{Kar}_0(\mathbb{M}(2)) \xleftarrow{j} \mathbb{M}(2) = \mathrm{Im}\phi \xleftarrow{\bar{\phi}} \mathbb{FM}(2).$$

This monoïd $\mathbb{M}(2)$ *is introduced by Lawvere as a* graphic monoïd.

Proof. In $\mathbb{G}(2)$ we get $\delta^2 = \delta, \gamma^2 = \gamma, \gamma\delta = \delta, \delta\gamma = \gamma$, and the full subcategory $\mathrm{End}_{\mathbb{G}}(\upsilon_1)$ of $\mathbb{G}(2)$ generated by υ_1 has one unit and two idempotents, γ and δ, which are splitted in $\mathbb{G}(2)$ as υ_0. If we denote by $\mathbb{M}(2) = \{1, c, d\}$ the monoïd with $c^2 = c, d^2 = d, cd = d, dc = c$, this monoïd is included in $\mathbb{G}(2)$

as $\mathrm{End}_{\mathbb{G}}(\upsilon_1)$ via $j : 1 \mapsto 1_{\upsilon_1}, c \mapsto \gamma, d \mapsto \delta$ and is a quotient of $\mathbb{FM}(2)$, the free monoïd on two generators c and d, via $\bar{\phi}$, with $\bar{\phi}(1_\upsilon) = 1$, $\bar{\phi}(c) = c$, $\bar{\phi}(d) = d$. We have $j.\bar{\phi} = \phi$.
Our $\mathbb{M}(2)$ is denoted Δ_1 and $\mathbf{M}(2)$ by Lawvere (see [9] and [10]), and a diagram of shape Δ_1 is named a *cylinder*. Lawvere observed that we recover $\mathbb{G}(2) = \mathrm{Kar}_0(\Delta_1)$ from the Cauchy completion $\mathrm{Kar}(\Delta_1) = \bar{\Delta}_1$ of the monoïd $\Delta_1 = \mathbb{M}(2)$ (the category obtained by splitting idempotents in Δ_1, also named "karoubian envelope" of Δ_1). As $\delta\gamma = \gamma$ and $\gamma\delta = \delta$, $\delta : (\upsilon_1, \delta) \to (\upsilon_1, \gamma)$ and $\gamma : (\upsilon_1, \gamma) \to (\upsilon_1, \delta)$ are morphisms between idempotents, and furthermore, for the same reason, they are inverse one of the other. We obtain $\mathrm{Kar}_0(\Delta_1)$ from $\mathrm{Kar}(\Delta_1)$ by reduction of these inverse isomorphisms to identities on one object υ_0. □

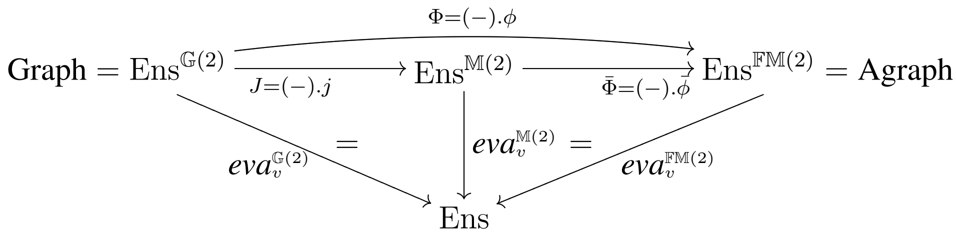
2. Autographic algebras

Burroni [3] defines a *graphic algebra* as an algebra of a monad on Graph. Similarly we define:

Definition 2.1. *An* autographic algebra *is an algebra of a monad on* Agraph*.*

2.1 Graphs are autographic algebras

Proposition 2.2. *In the following diagram, all the functors are monadic:*



Especially, graphs are autographic algebras.

Proof. It is easy to show that J is an equivalence of categories [1, ex. 3.4, p.107]. This comes from the fact that splitting idempotents is an absolute limit construction. Then we use the known fact that for any monoïd M the forgetful functor $\mathrm{Ens}^M \to \mathrm{Ens}$ is monadic [1, ex. 3.5, p.109], and we get that the three evaluations are monadic. These facts could be proved by the Linton characterization of monadicity over Ens, and the property for $\bar{\Phi}$ by its extension by Borceux and Day [2] for monadicity over a category of presheaves. □

In the next sections we will need two criterions of monadicity which could have been used here.

Proposition 2.3. *The proposition 2.2 could also be proved using Coppey or Lair criterions of monadicity.*

Proof. 1 — According to the *Coppey's criterion* [4, Prop. 2, p.17], the monadicity of a functor $\mathrm{Ens}^{\mathbb{D}} \xrightarrow{\mathrm{Ens}^K} \mathrm{Ens}^{\mathbb{C}}$ for $K : \mathbb{C} \to \mathbb{D}$ a functor bijective on objects, is equivalent to the existence of a left adjoint.
This works especially for any morphism of monoïds $f : M' \longrightarrow M$, and so here for $\bar{\phi} : \mathbb{FM}(2) \longrightarrow \mathbb{M}(2)$.

2 — The *Lair's criterion* [7, thm.2, p.278] [8, Corollaire, p.8]. says that the VTT condition of Beck [11, Th. 1, p147, ex. 6, p. 151] for tripleability is satisfied for a projectively sketched functor $U = \mathrm{Ens}^K : \mathrm{Ens}^{S'} \to \mathrm{Ens}^S$, sketched by a morphism of projective sketches $K : S \to S'$ if K is *basic* (or *of 'Kleisli'*), i.e. if any distinguished cone in S' is based in S, and any new object in S' is the top of a distinguished cone in S'.
Here this criterion can be applied to $\bar{\phi} : \mathbb{FM}(2) \to \mathbb{M}(2)$ considered as a morphism of projective sketches, with no cones, and with no new object in $\mathbb{M}(2)$, or it could be applied to $\phi : \mathbb{FM}(2) \to \mathbb{G}(2)$, with, as a distinguished cone, the one specifying the new object v_0 as a kernel, based in $\mathbb{M}(2)$.

□

2.2 Basic graphic algebras are autographic algebras

Proposition 2.4. *If* $W : \mathcal{X} \to$ Graph *is algebraic, i.e. if, via* W*,* $\mathcal{X}$ *is a category of* graphic algebras *in the sense of Burroni [3], and if, more strictly,* W *is sketched by a basic morphism of small projective sketches (in the sense of Lair)* $K : \mathbb{G}(2) \to S'$*, with* $\mathcal{X} = \mathrm{Set}^{S'}$ *and* $W = \mathrm{Set}^K$*, then the functor* $\mathrm{Set}^{K\phi} : \mathcal{X} \to$ Agraph *is algebraic, i.e., via* $\mathrm{Set}^{K\phi}$*,* $\mathcal{X}$ *is a category of* autographic algebras*. For example this works for* $\mathcal{X} =$ Cat*: categories are autographic algebras.*

Proof. As j induces an equivalence, the question is reduced to the transfer of monadicity from $\mathrm{Set}^{\mathbb{M}(2)}$ to $\mathrm{Set}^{\mathbb{FM}(2)}$, via $\mathrm{Set}^{\bar{\phi}}$. At first let us recall that this functor is monadic, i.e. that the proposition is valid if $W = \mathrm{Id}_{\mathrm{Graph}}$. But as $K : \mathbb{G}(2) \to S'$ is basic (see definition in proposition 2.3), also $Kj : \mathbb{M}(2) \to \mathbb{S}'$ is basic, and then $Kj\bar{\phi} = K\phi : \mathbb{M}(2) \to S'$ is basic. So $\mathcal{X}$ is a category of (basic) autographic algebras. □

2.3 Autographs in Lawvere algebras are autographic algebras

Proposition 2.5. *Let T be the sketch of a Lawvere theory. Then the category* $\mathrm{Agraph}\big(\mathrm{Set}^T\big)$ *of autographs in* Set^T *is monadic over* Agraph.

Proof. Let u be the object in T such that each objects is specified as a u^n, and let $K : \mathbb{FM}(2) \to \mathbb{FM}(2) \times T : m \mapsto (m, u)$. This functor is basic, and so the functor $\mathrm{Set}^K : \mathrm{Set}^{\mathbb{FM}(2)\times T} \to \mathrm{Set}^{\mathbb{FM}(2)} = \mathrm{Agraph}$ is monadic. And $\mathrm{Agraph}(\mathrm{Set}^T) = \big(\mathrm{Set}^T\big)^{\mathbb{FM}(2)} = \mathrm{Set}^{T\times\mathbb{FM}(2)}$. □

2.4 Quotients of $\mathbb{FM}(2)$ and reflexive subcategories of Agraph

Proposition 2.6. *Any presentation of a* 2*-generated monoïd* M*, i.e. any quotient map of monoïds* $q_M : \mathbb{FM}(2) \to M$ *determines by composition on the right a functor* $\mathrm{Set}^{q_M} : \mathrm{Set}^M \to \mathrm{Set}^{\mathbb{FM}(2)}$ *with left and right adjoints* $\mathrm{Lan}_{q_M} \dashv (-)^{q_M} \dashv \mathrm{Ran}_{q_M}$*, and the adjunction* $\mathrm{Lan}_{q_M} \dashv (-)^{q_M}$ *determines the topos* Set^M *as a reflexive subcategory of* Agraph *which is a category of autographic algebras. Especially this works for* M *the monoïds* $\mathbb{FIM}(2), \mathbb{FGM}(2), \mathbb{FSM}(2) = \mathbb{M}(2)$ *given in Prop. 2.8.*

Proof. As in Prop. 2.3 it is a consequence of [4, Prop. 2, p.17]. Let us precise that the corresponding idempotent monad $T_M = (-)^{q_M}\mathrm{Lan}_{q_M}$ is clear; for (A, d, c) in $\mathrm{Set}^{\mathbb{FM}(2)}$ we have $T_M(A, d, c) = A/[q_M]$, with $[q_M]$ the smallest congruence on (A, d, c) such that

$$\forall m, m' \in \mathbb{FM}(2)\,\forall u \in A\,\big(q_M(m) = q_M(m') \Rightarrow mu = m'u \mod [q_M]\big);$$

so, for any $x, y \in A$, we have $x = y \mod [q_M]$ if and only if

$$\begin{array}{l}\exists k \geq 1, \forall j \leq k, \exists m_j, m'_j \in \mathbb{FM}(2), q(m_j) = q(m'_j), \exists u_j \in E, \\ x = m_1u_1,\ m'_1u_1 = m_2u_2, \ldots m'_{k-1}u_{k-1} = m_ku_k,\ m'_ku_k = y.\end{array}$$

□

An example of Prop. 2.6 is:

Proposition 2.7. *With* $\mathbb{FM}(1) = \mathbb{N}$ *(equipped with* $+$*) the free monoïd on one generator,* $\mathrm{Set}^{\mathbb{FM}(2)}$ *is the pullback of* $\mathrm{Set}^{\mathbb{FM}(1)} \to \mathrm{Set}$ *with itself. The topos* $\mathrm{Set}^{\mathbb{FM}(1)}$ *of "primary structures" is an algebraic category over the topos* $\mathrm{Set}^{\mathbb{FM}(2)}$ *of autographs: primary structures are autographic algebras.*

Proof. The objects of $\mathrm{Set}^{\mathbb{FM}(1)}$ are named *primary structures* by M. Lazard. Here we have a quotient map of monoïds

$$q_1 : \mathbb{FM}(2) \to \mathbb{FM}(2)/(c = d) = \mathbb{FM}(1),$$

and so (Prop. 2.6) $\mathrm{Set}^{q_1} : \mathrm{Set}^{\mathbb{FM}(1)} \to \mathrm{Set}^{\mathbb{FM}(2)}$ is monadic. □

2.5 2-generated graphic topos

One example of Prop.2.6 is associated to the free 2-generated graphic monoïd.

Proposition 2.8. *1 — The monoïd* $\mathbb{M}(2)$ *is the free* right singular monoïd *(in which* $xy = y$*) on* 2 *generators* c *and* d*, and so is denoted by* $\mathbb{FSM}(2)$*.*
2 — The monoïd $\mathbb{M}(2)$ *is a* right graphic monoid *(in which* $xyx = yx$*), but it is not a free one.*
3 — The free right graphic monoid *on* 2 *generators* c *and* d *is denoted by* $\mathbb{FGM}(2)$*. It has* 5 *elements.*
4 — $\mathbb{G}(2) = \mathrm{Kar}_0(\mathbb{M}(2))$ *is a quotient of* $\bar{\mathbb{G}}(2) = \mathrm{Kar}_0(\mathbb{FGM}(2))$*.*
5 — The monoïd $\mathbb{FGM}(2)$ *is an* idempotent monoïd *(in which* $x^2 = x$*), but it is not a free one.*
6 — The free idempotent monoïd *on* 2 *generators* c *and* d *is denoted by* $\mathbb{FIM}(2)$*. It has* 7 *elements. A quotient of* $\mathbb{FIM}(2)$ *is* $\mathbb{FGM}(2)$*.*

Proof. $\mathbb{FGM}(2) = \{1, c, d, cd, dc\}$ is obtained by adjunction of a unit to the free right regular semigroup on two generators c and d. Then $\mathbb{M}(2)$ is a quotient of $\mathbb{FGM}(2)$, with $cdc = dc$ and $dcd = cd$. The notion of a *graphic monoïd* M is used by Lawvere to introduce *graphic toposes* $\mathrm{Ens}^{M^{\mathrm{op}}}$. A *band* is a semigroup where every element is idempotent, a left regular band is a band with $xyx = xy$, and so a graphic monoïd according to Lawvere is exactly a left regular band with unit. Then an explicit construction of the free left regular band on n generators is given in [6], which for $n = 2$ gives

a semigroup with 4 elements, and with one more element as unit we get $\mathbb{FGM}(2)$.
We have $\mathbb{FIM}(2) = \{1, c, d, cd, dc, cdc, dcd\}$. The description of $\mathbb{FIM}(2)$ is given in [12]. □

Proposition 2.9. *Any graphic topos* $\mathrm{Set}^{\mathrm{Kar}_0(M)} \simeq \mathrm{Set}^M$ *with* M *a* 2*-generated graphic monoïd is a category of autographic algebras.*

Proof. Any such M is a quotient of $\mathbb{FGM}(2)$, and so of $\mathbb{FM}(2)$. Of course as $\mathbb{FIM}(2)$ is finite, there is only a finite number of such M. □

3. Autocategories, associative autographs, flexicategories

The following definitions precise the situation of autocategories between associative autographs and flexicategories. We see that associative autographs and autocategories are examples of autographic algebras.

Definition 3.1. *An* associative autograph $\mathcal{A}$ *is an autograph* (A, d, c) *equipped with* composition $gf : p \to r$ *for any* $f : p \to q$, $g : q \to r$*, such that the two compositions of three consecutive arrows are equal:*

$$\text{Associativity:} \quad h(gf) = (hg)f, \ \text{ if } \ f : p \to q,\ g : q \to r,\ h : r \to s.$$

An associative autograph $\mathcal{A}$ *is* unitary *if the underlying autograph* (A, d, c) *is with* identifiers *— i.e. with specified arrows* $i_{df} : df \to df$, $i_{cf} : cf \to cf$ *for every* $f \in A$ *— and if these identifiers are identities i.e. units for composition:*

$$\text{Unitarity:} \quad fi_{df} = f = i_{cf}f.$$

An unitary associative autograph *is shortly named an* autocategory *([5, p.76]).*

The category of associative autographs is denoted by AAgraph (with morphisms the maps $F : A \to A'$ with $d'Fa = Fda$, $c'Fa = Fca$, $F(ba) = F(b)F(a)$ if $db = ca$).
The forgetful functor $\mathrm{W} : \mathrm{AAgraph} \to \mathrm{Agraph}$ is given by $\mathrm{W}(\mathcal{A}) = (A, d, c)$.
The category of autocategories is denoted by Acat: it is the full subcategory of AAgraph with objects the autocategories. We have a forgetful functor $\mathrm{W}' = WI : \mathrm{Acat} \to \mathrm{Agraph}$, with $I : \mathrm{Acat} \to \mathrm{AAgraph}$ the inclusion functor.

Remark 3.2. In the definition of an autocategory, identities are unique, and so "unitary" is a property of an associative autograph, and not a supplementary data on it. So autocategories are to associative autographs as monoïds are to semigroups.

Definition 3.3. *A* flexicategory *is a category $\mathcal{C}$ equipped with a* flex *i.e. a map $\varphi : \text{Obj}(\mathcal{C}) \to \text{Arrow}(\mathcal{C})$. The category of flexicategories is* Fcat*, with morphisms functors $F : \mathcal{C} \to \mathcal{C}'$ such that $\varphi'(F(X)) = F(\varphi(X))$.*

We have a forgetful functor $W'' : \text{Fcat} \to \text{Cat}$ given by $W''((\mathcal{C}, \varphi)) = \mathcal{C}$.

Proposition 3.4. *An autocategory $\mathcal{A}$ "is" a mono-flexicategory , i.e. a category $U\mathcal{A} = \mathcal{C}$ equipped with a* flex *$\varphi : \text{Obj}(\mathcal{C}) \to \text{Arrow}(\mathcal{C})$, with the condition that φ is injective. So is defined an inclusion $J : \text{Acat} \to \text{Fcat}$, and we have the forgetful functor $\text{W}''' = W''J : \text{Acat} \to \text{Cat}$.*

Proof. In [5, Prop.6.2., p.76] we saw that any mono-flexicategory determines an autocategory. But conversely, given an autocategory $\mathcal{A}$ we can introduce an underlying category $\mathcal{C}$ with objects the O_u, with u any identifier $u = i_{df}$ or $u = i_{cf}$ in $\mathcal{A}$, and then sources and targets are given by $s(f) = O_{df}$, $t(f) = O_{cf}$, and the identities on objects are $\text{Id}_{O_u} = u$. □

Proposition 3.5. *1 — The category of autocategories is a category of autographic algebras, the associated monad is the construction of "paths with identity" given in [5, Prop. 6.3, p. 77]*

$$\text{P}^{\iota}(A, (d, c)) = \big(\text{Path}^{\iota}(A, (d, c)), D, C\big).$$

2 — The category AAgraph *of associative autographs is a category of autographic algebras, the associated monad is the construction of "paths without adding identities" given by*

$$\text{P}(A, (d, c)) = \big(\text{Path}(A, (d, c)), D, C\big),$$

analogous to $\text{P}^{\iota}(A, (d, c))$ excepted that we do not add identities.

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