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The Locally Free Relatively Filtered Diagram as an Inductive Completion of a System of Choice*

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Abstract. Guitart and Lair [5] have established the existence of Locally Free Diagrams, which can be seen as a purely categorical version of the solution set condition, and of the Lowenheim–Skolem theorem. Their proof is based on a transfinite construction by saturation. An iterative principle is established, but the construction is not effective for every step. The thesis of Gerner [3] contains a more effective proof for the existence of Locally Free Diagrams (with the restriction that the projective bases of the sketch S must all be finite). But the problem of [3] lies in the impossibility to name concretely the elements of the Locally Free Diagrams. The present paper will provide a new construction of the Locally Free Diagram in which the effective and the non-effective part will be much more separated (again the projective bases must all be finite). This construction represents a notable improvement with regard to the proof of [3] allowing the concrete designation of the elements of the Locally Free Diagrams. Furthermore we show that the construction is relatively filtered (i.e. satisfies the "filtered"-property).

Key words: sketches, completions, free structures.

1. Prerequisites

1.1. Motivation

1.1.1. *Free Structures*

In Algebra, free structures have been studied with interest for some time: the free monoid generated by an alphabet, the abelian group generated by a set, etc. In all these cases the situation is the same: there is a set X on which we want to construct an algebraic structure of a given type such that for any function from X to an algebraic structure M of this type there is a unique factorization property for the free algebraic structure $F(X)$ generated by X. We can express this with the following formula:

$$Hom(X, M) \cong Hom(F(X), M).$$

In the case of the monoids we can effectively construct the elements of the free monoid generated by an alphabet as the words on this alphabet.

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1.1.2. *Multi-Free Structures*

Following this, there are other "algebraic" structures, e.g. the fields, which do not provide unique free structures. (It is not possible to generate the free field on a set.) For this reason we should generalize the notion of a "free structure" and so we obtain the notion of a *multi-free structure*. Instead of a unique factorization by a unique free structure, we have a set of free structures and a unique factorization by one of these free structures which is also unique. This is the case that Diers [1] considered. The fields are an example for such multi-free structures. We can express this with the following formula:

$$Hom(X, M) \cong \coprod_{A \in \underline{A}(X)} Hom(F_A(X), M).$$

1.1.3. *Locally Free Diagrams*

Finally there is the possibility that we can find a factorization by a free structure among several free structures, but this is not the unique factorization (for example we can refer to epimorphisms). In this case, where the free structures are neither indexed by the singleton set nor by any arbitrary set but by a small category. The structures obtained thus, are called *Locally Free Diagrams* (L.F.Ds.). This notion was introduced in [5]. The factorization property of L.F.Ds. can be expressed by the following formula:

$$Hom(X, M) \cong \varinjlim_{A \in \underline{A}(X)} Hom(F_A(X), M).$$

1.2. Definitions

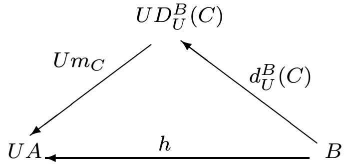
In all generality, we now can precisely define the notion of an L.F.D.:

DEFINITION 1.2.1. Let $\underline{A}$ be a category, $\underline{B}$ a complete and co-complete category and $U\colon \underline{A} \to \underline{B}$ a functor. The functor U admits *Locally Free Diagrams* (L.F.Ds.) if, and only if, for any $B \in \underline{B}$ there is a category $LFD_U(B) \in ObCat$, a functor $D_U^B\colon LFD_U(B) \to \underline{A}$ and a projective cone $d_U^B = (B \to UD_U^B(C))_{C \in LFD_U(B)}$ such that for any $A \in \underline{A}$:

$$Hom_{\underline{B}}(B, UA) \cong \varinjlim_{C \in LFD_U(B)} Hom_{\underline{A}}(D_U^B(C), A).$$

This last condition can be expressed more explicitly: For any arrow $(B \xrightarrow{h} UA)$ in $\underline{B}$:

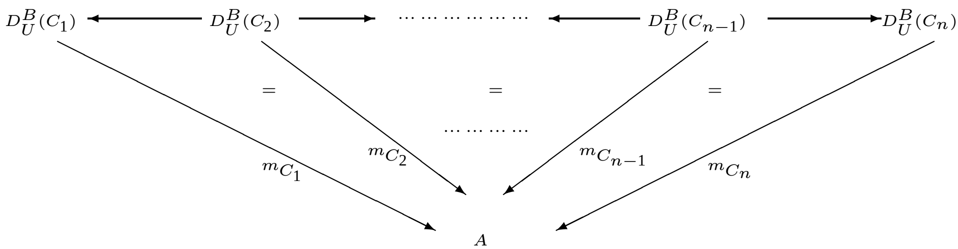
(1) $\exists C \in LFD_U(B)$, $\exists (D_U^B(C) \xrightarrow{m_C} A) \in Hom_{\underline{B}}(D_U^B(C), A)$: $h = Um_C \circ d_U^B(C)$



(2) If (C, m_C) and $(C', m_{C'})$ satisfy the condition (1), then there is a zig-zag in $LFD_U(B)$:

$$(C = C_1 \leftarrow C_2 \rightarrow \cdots \leftarrow C_{n-1} \rightarrow C_n = C')$$

such that the following diagram is commutative:



REMARK 1.2.2. The limit property of the Hom can also be expressed in the following way: If $\mathcal{H}$ is the functor $LFD_U(B) \to (B \downarrow U)$ with $C \mapsto (d_U^B(C)\colon B \to UD_U^B(C))$, then the diagram $(LFD_U(B), D_U^B, d_U^B)$ is locally free, when for any arrow $(B \xrightarrow{h} UA)$ in $\underline{B}$ the comma-category $\mathcal{H} \downarrow h$ is non-empty and connected.

Let us provide two examples of L.F.Ds. (the first diagram is a multi-free structure, whereas in the second case we have a "general" L.F.D.).

EXAMPLE 1.2.3. Let $\underline{A}$ be the category of commutative fields, $\underline{B}$ the category of commutative rings, $U\colon \underline{A} \to \underline{B}$ the inclusion functor and $\mathbf{Z} \in Ob\underline{A}$ the ring of integers. Then U admits on $\mathbf{Z}$ the following multi-free structure (indexed by the set of prime numbers): $\{\mathbf{Q}, \mathbf{Z}/_{p\mathbf{Z}}\}$.

EXAMPLE 1.2.4. Let $\underline{A}$ be the category of commutative rings whose characteristics are strictly positive, $\underline{B}$ the category of commutative rings, and $U\colon \underline{A} \to \underline{B}$ the inclusion functor. Since the ring of integers $\mathbf{Z}$ has characteristic 0, $\mathbf{Z}$ is not in $\underline{A}$, but it is possible to generate an L.F.D. for $\mathbf{Z}$ in the following way: If $h\colon \mathbf{Z} \to A$ is a ring morphism from $\mathbf{Z}$ to a ring A of characteristic n with $n > 0$, clearly, we can factorize h by $\mathbf{Z}/_{n\mathbf{Z}}$ and that if $\mathbf{Z}/_{n\mathbf{Z}}$ and $\mathbf{Z}/_{m\mathbf{Z}}$ factorize both the morphism h, then either $n|m$ or either $m|n$. So the underlying category of the L.F.D. will be $\mathbf{N}^*$ with arrows $n \to m$ when $m|n$.

It is natural to ask whether L.F.Ds. exist, and if they exist, whether they are unique. As we mentioned before, in [5], the existence of L.F.Ds. has been established for a sketchable functor U, particularly for $\underline{A} = Mod[S]$, $\underline{B} = Set^{\underline{C}}$ such that U is the full and faithful inclusion functor and such that $S = (\underline{C}, \mathbb{I}, \mathbb{P})$ is a mixed sketch whose sets of inductive and projective cones can be noted thus:

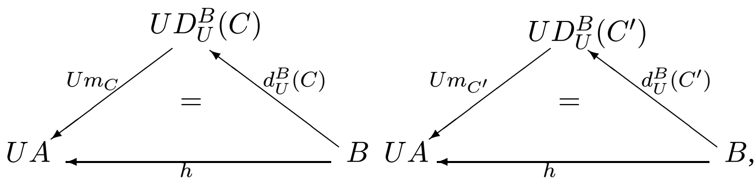
– $\mathbb{I} = \{I = (U^I_J \xrightarrow{\alpha^I_J} U^I)_{J \in \underline{J}^I}$ such that $\underline{J}^I$ is a small category$\}$,

– $\mathbb{P} = \{P = (V^P \xrightarrow{\beta^P_K} V^P_K)_{K \in \underline{K}^P}$ such that $\underline{K}^P$ is a small category$\}$.

Unfortunately, L.F.Ds. are not unique at all. However it is possible to apply the notion of a filtered category to L.F.Ds. and for the L.F.Ds. which satisfy the "filtered"-property (we call them *Locally Free Relatively Filtered Diagrams*) it is possible to show their uniqueness up to an homotopy-equivalence of their associated classifying spaces. This has been shown in Guitart [6] and Gerner [3] provided a one step construction of L.F.R.F.Ds.

DEFINITION 1.2.5. The locally free diagram $(LFD_U(B), D^B_U, d^B_U)$ is *relatively filtered* (L.F.R.F.D.), when for any arrow $(B \xrightarrow{h} UA)$ in $\underline{B}$ the comma-category $\mathcal{H} \downarrow h$, which is non-empty and connected, is filtered [7] even more. The word "relatively" refers to h. This condition can be expressed more explicitly:

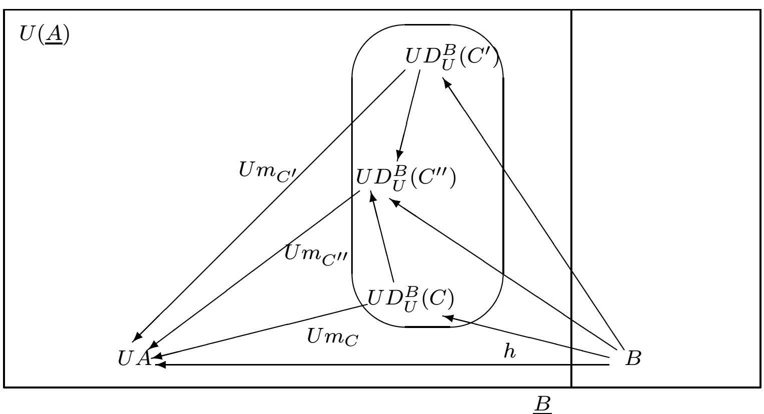
– For any object $A \in \underline{A}$ and any morphism $(B \xrightarrow{h} UA) \in Arr\underline{B}$, for $C, C' \in ObLFD_U(B)$ and $m_C \in Hom_{\underline{B}}(D^B_U(C), A)$ and $m'_C \in Hom_{\underline{B}}(D^B_U(C'), A)$ such that the following diagrams are commutative:



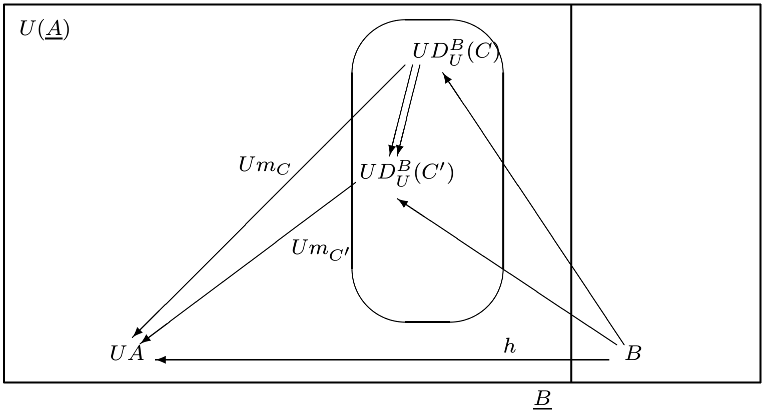
there are $C'' \in ObLFD_U(B)$, $m_{C''} \in Hom_{\underline{B}}(D^B_U(C''), A)$, and

$C \to C'' \leftarrow C'$

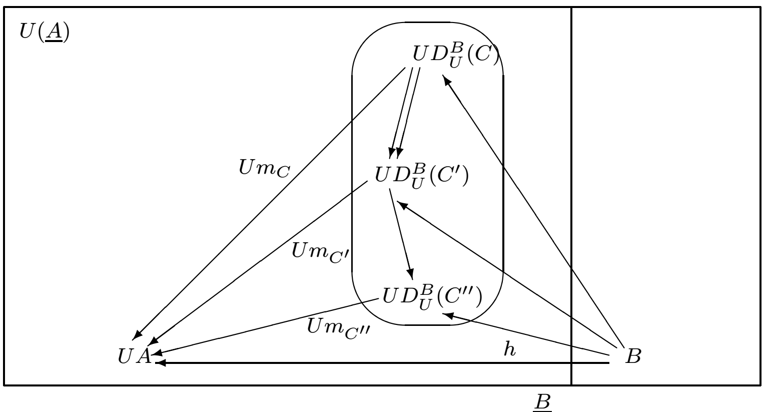
in $ArrLFD_U(B)$ such that in $\underline{B}$ we have a commutative diagram:



– For any object $A \in \underline{A}$ and any morphism $(B \xrightarrow{h} UA) \in Arr\underline{B}$, for $C, C' \in ObLFD_U(B)$ and any pair of parallel arrows u: $C \to C'$ and v: $C \to C'$, for $m_C \in Hom_{\underline{B}}(D_U^B(C), A)$ and for $m_{C'} \in Hom_{\underline{B}}(D_U^B(C'), A)$ such that the following diagram is commutative:



there is $C'' \in ObLFD_U(B)$, $m''_C \in Hom_{\underline{B}}(D_U^B(C''), A)$ and w: $C' \to C''$ in $ArrLFD_U(B)$ such that $w \circ u = w \circ v$ and such that the following diagram is commutative in $\underline{B}$:



As mentioned above, we now have the following kind of “uniqueness” result:

THEOREM 1.2.6. *The categories $LFD_U(B)$ and $B \downarrow U$ are homotopically equivalent.*

Proof. This result which uses Theorem A of Quillen [8] is due to Guitart [6]. □

COROLLARY 1.2.7. *Two L.F.R.F.Ds. have the same homotopy type.*

We now return to the problem of existence of L.F.Ds. (L.F.R.F.Ds.). After the existence result of [5], in [3] is presented a more effective existence proof of L.F.Ds. and of L.F.R.F.Ds. (again for a sketchable functor U and with the restriction that the projective bases of the sketch must be all finite). In Section 2 we show a new existence proof of L.F.Ds. (again for a sketchable functor U and with the restriction that the projective bases of the sketch must be all finite) which will be even easier to use because it separates much more than before what is effective from what is ineffective. Since this new proof partially coincides with the proof of Gerner, we will very often refer to [3] or to [4] in Section 2.

2. A New Constructive Proof for the Existence of L.F.Ds.

2.1. Motivation

If a functor $(\underline{C} \xrightarrow{T} Set) \in Set^{\underline{C}}$ is not a model (i.e. is not in $Mod[S]$), there are three principal reasons for this (the first two concern inductive limits, the third refers to projective limits).

ERROR 1. It is possible that for an inductive cone of S the image of its vertex under T is not completely reached by the base, i.e. $\exists I = (U^I_J \xrightarrow{\alpha^I_J} U^I)_{J \in \underline{J}^I} \in \mathbb{I}$ such that $\varinjlim_{J \in \underline{J}^I} T(U^I_J) \neq T(U^I)$ because $\bigcup_{J \in \underline{J}^I} T(U^I_J) \subsetneq T(U^I)$.

ERROR 2. It is possible for an inductive cone of S that in the image of its vertex under T two points of the base are identified without there being a zig-zag in the base to join them together, i.e. there is $I = (U^I_J \xrightarrow{\alpha^I_J} U^I)_{J \in \underline{J}^I} \in \mathbb{I}$ such that $\varinjlim_{J \in \underline{J}^I} T(U^I_J) \neq T(U^I)$ because there are $J, J' \in \underline{J}^I$, $x_J \in T(U^I_J)$ and $x_{J'} \in T(U^I_{J'})$ with $(T(\alpha^I_J)(x_J) = T(\alpha^I_{J'})(x_{J'}))$ but without any possibility of joining (x_J, J) and $(x_{J'}, J')$ together by a zig-zag.

ERROR 3. There is $P = (V^P \xrightarrow{\beta^P_K} V^P_K)_{K \in \underline{K}^P} \in \mathbb{P}$ such that $\varprojlim_{K \in \underline{K}^P} T(V^P_K) \neq T(V^P)$.

We try to correct these three error-types by convenient saturations of “error-sets”. However, as we achieve all these saturations simultaneously, the resulting functor will again contain the three errors mentioned above. So we should start the same procedure again. If T_n is the nth functor of this procedure, we shall denote $\coprod_{n \in \mathbf{N}} T_n$ by T_∞ and quotient it by a convenient equivalence relation which will give a model $T_\infty/_\approx$. Therefore, such a model $T_\infty/_\approx$ is given by the choice of a convenient infinite saturation string and of an equivalence relation. For any of these infinite saturation strings, considered without the convenient equivalence relation, we can start again the same procedure of saturation and find other infinite saturation strings depending not on T but on T_∞. We thus obtain functors of the type $T_{\infty\infty}$. By induction we can prolong this procedure and construct finite words of infinite saturation strings where any letter depends on the preceeding one. These finite words together with the convenient equivalence relations, will be the set of objects of the underlying category of the L.F.D. of T (see the exact definition in Section 2.2). But first let us define the basic saturation-strings.

2.2. The pointwise construction of the L.F.D.

2.2.1. *The Basic Saturation Strings*

Let $h\colon T \to M$ be a morphism from T to a model M of S. We shall construct a set of saturations and Gerner shows in [3] that there are saturations in this set which correct the errors 1, 2 and 3 and which transform T into a model of the sketch S. Hence the definition of the saturation set depends on errors 1, 2 and 3:

ERROR 1. For any inductive cone I of the sketch S we define $T^I = T(U^I) \setminus [\bigcup_{J\in \underline{J}^I} T(\alpha^I_J)(T(U^I_J))]$ as the subset of points of the vertex $T(U^I)$ which are not reached by points of the base $(T(U^I_J))_{J\in\underline{J}^I}$. We can define the repair-measure of error 1 by

$$\Lambda(\underline{J}^I, T^I) = \{\lambda^I\colon\ Ob\underline{J}^I \to \mathcal{P}(T^I)/\lambda^I \text{ is a map}\}.$$

ERROR 2. Here we must try to correct the problem of “missing zig-zags” in the base $(T(U^I_J))_{J\in\underline{J}^I}$. Without restriction to generality we will only consider zig-zags of the following shapes:

$$\begin{array}{l} J' \longrightarrow J \longleftarrow J'' \\ J' \xleftarrow{\gamma} J \xrightarrow{\delta} J'' \end{array}$$

and only the second type of zig-zag $J' \xleftarrow{\gamma} J \xrightarrow{\delta} J''$ will represent a problem of “missing zig-zags” as it is shown in [3]. Let us define

$$\begin{aligned}\Theta^{\gamma,\delta}(J) \ = \ &\{(x_{J'}, x_{J''}) \in T(U^I_{J'}) \times T(U^I_{J''}) \text{ such that} \\ &\forall x_J \in T(U^I_J)\colon\ T(\gamma)(x_J) \neq x_{J'} \text{ or } T(\delta) \neq x_{J''}\}.\end{aligned}$$

With $\Theta^I(J) := \coprod_{\gamma,\delta} \Theta^{\gamma,\delta}(J)$ we are now able to define the repair-measure of error 2 by

$$\Theta(\underline{J}^I, T) = \{\vartheta^I\colon\ Ob\underline{J}^I \to \mathcal{P}(\Theta^I)/\ \forall J \in \underline{J}^I\colon\ \vartheta^I(J) \subseteq \Theta^I(J)\}$$

where $\Theta^I = \coprod_{J\in\underline{J}^I} \Theta^I(J)$.

ERROR 3. In order to control the problem of T not transforming projective cones of S into projective limits of Set, we shall define for any projective cone $P \in \mathbb{P}$ of S:

$$\Psi(\underline{K}^P, T) \overset{\text{def}}{=} \mathcal{P}\Big(\prod_{K\in\underline{K}^P} T(V^P_K)\Big).$$

The global measure of the attempt to correct errors 1, 2 and 3 is the following set:

$$\Delta_0 = \Delta_0(T) = \prod_{I\in\mathbb{I}} \Lambda(\underline{J}^I, T) \times \prod_{I\in\mathbb{I}} \Theta(\underline{J}^I, T) \times \prod_{P\in\mathbb{P}} \Psi(\underline{K}^P, T).$$

For any element $(\lambda, \vartheta, \psi) \in \Gamma$ we now have the following "saturation-functor" $T_1 = T_1(\lambda, \vartheta, \psi)$:

$$\begin{aligned}
T_1(\lambda,\vartheta,\psi)\colon\ \underline{C} &\rightarrow Set\\
Ob\underline{C} &\rightarrow ObSet\\
W &\mapsto T(W) + \coprod_{I\in\mathbb{I}} \coprod_{J\in\underline{J}^I} Hom_{\underline{C}}(U^I_J, W) \times \lambda^I(J) \text{ (error 1)}\\
&\quad + \coprod_{I\in\mathbb{I}} \coprod_{J\in\underline{J}^I} Hom_{\underline{C}}(U^I_J, W) \times \vartheta^I(J) \text{ (error 2)}\\
&\quad + \coprod_{P\in\mathbb{P}} Hom_{\underline{C}}(V^P, W) \times \psi^P \text{ (error 3)}\\
Arr\underline{C} &\rightarrow ArrSet\\
(W \xrightarrow{\varepsilon} W') &\mapsto (T_1(\lambda,\vartheta,\psi)(W) \xrightarrow{T_1(\varepsilon)} T_1(\lambda,\vartheta,\psi)(W'))
\end{aligned}$$

$T_1(\varepsilon)$ is defined piecewisely:

$$\begin{aligned}
T(W) &\xrightarrow{T(\varepsilon)} T(W')\\
Hom_{\underline{C}}(U^I_J, W) \times \lambda^I(J) &\longrightarrow Hom_{\underline{C}}(U^I_J, W') \times \lambda^I(J)\\
(U^I_J \xrightarrow{\theta} W, x) &\longmapsto (U^I_J \xrightarrow{\theta} W \xrightarrow{\varepsilon} W', x)\\
Hom_{\underline{C}}(U^I_J, W) \times \vartheta^I(J) &\longrightarrow Hom_{\underline{C}}(U^I_J, W') \times \vartheta^I(J)\\
(U^I_J \xrightarrow{\theta} W, x) &\longmapsto (U^I_J \xrightarrow{\theta} W \xrightarrow{\varepsilon} W', x)\\
Hom_{\underline{C}}(V^P, W) \times \psi^P &\longrightarrow Hom_{\underline{C}}(V^P, W') \times \psi^P\\
(V^P \xrightarrow{\theta} W, x) &\longmapsto (V^P \xrightarrow{\theta} W \xrightarrow{\varepsilon} W', x).
\end{aligned}$$

However, it is possible that the different saturations in the functor T_1 are each disrupting the effect of the others. Therefore we should restart the same procedure described above for the functor T_1. Hence we obtain the set

$$\Gamma_1 = \Gamma_1(\lambda, \vartheta, \psi) = \prod_{I\in\mathbb{I}} \Lambda(\underline{J}^I, T_1) \times \prod_{I\in\mathbb{I}} \Theta(\underline{J}^I, T_1) \times \prod_{K\in\underline{K}^P} \Psi(\underline{K}^P, T_1).$$

So we can define $\Delta_1 = \coprod_{(\lambda,\vartheta,\psi)\in\Gamma} \Gamma_1(\lambda, \vartheta, \psi)$. This procedure can now be prolonged by infinite induction. Hence we obtain functors $T_1, \ldots, T_n$ and sets $\Gamma_1, \ldots, \Gamma_n$; $\Delta_1 = \Delta_1(T), \ldots, \Delta_n = \Delta_n(T)$.

2.2.2. *The Construction of the L.F.D.*

Let us define

$$\Delta_\infty = \Delta_\infty(T) \stackrel{\text{def}}{=} \Big\{\rho\colon \mathbb{N} \to \coprod_{n\in\mathbb{N}} \Delta_n(T)\Big\}.$$

For $\rho \in \Delta_\infty$ we can define the following saturation functor:

$$T(\rho) \stackrel{\text{def}}{=} T + \coprod_{n\in\mathbb{N}} T_{l(\rho(n))+1}(\rho(n)),$$

where $l(\rho(n)) \in \mathbb{N}$ such that $\rho(n) \in \Delta_{l(\rho(n))}$.

By induction we can define now the finite saturation words $(\rho_1, \dots, \rho_n)$ where each member is defined on the basis of the preceeding step: For $n = 1$, $\rho_1 \in \Delta_\infty(T)$. Suppose next that we have defined a string of shape $(\rho_1, \dots, \rho_n)$ and a functor $T(\rho_1, \dots, \rho_n) = T(\rho_1)(\rho_2)\cdots(\rho_n)$. For $\rho \in \Delta_\infty(T(\rho_1, \dots, \rho_n))$ we can consider the string $(\rho_1, \dots, \rho_n, \rho)$ and the functor $T(\rho_1, \dots, \rho_n, \rho) \stackrel{\text{def}}{=} T(\rho_1, \dots, \rho_n)(\rho)$. Let us define

$$\begin{aligned}\Delta^*_\infty &= \Delta^*_\infty(T)\\ &\stackrel{\text{def}}{=} \{(\rho_1, \dots, \rho_n) | \forall i \in \{1, \dots, n\}\colon\ \rho_i \in \Delta_\infty(T(\rho_1, \dots, \rho_{i-1}))\}.\end{aligned}$$

For $(\rho_1, \dots, \rho_n) \in \Delta^*_\infty$, for $H \in Mod[S]$, for $g\colon\ T(\rho_1, \dots, \rho_n) \to H$ and for $C \in \underline{C}$ we have an equivalence relation $\approx_{g(C)}$:

$$\forall x, y \in T(\rho_1, \dots, \rho_n)(C)\colon\ x \approx_{g(C)} y \Leftrightarrow g(C)(x) = g(C)(y).$$

Therefore we can define for $(\rho_1, \dots, \rho_n) \in \Delta^*_\infty$ the following class:

$$\begin{aligned}\Sigma &= \Sigma(\rho_1, \dots, \rho_n) = \Sigma(T, (\rho_1, \dots, \rho_n))\\ &\stackrel{\text{def}}{=} \{g\colon\ T(\rho_1, \dots, \rho_n) \to H/H \in ObMod[S],\\ &\qquad T(\rho_1, \dots, \rho_n)/_{\approx g} \text{ is a model of } S\}.\end{aligned}$$

The class Σ becomes a set if we quotient Σ by the following equivalence relation which is defined for all $T(\rho_1, \dots, \rho_n) \stackrel{g}{\longrightarrow} H$ and $T(\rho_1, \dots, \rho_n) \stackrel{g'}{\longrightarrow} H' \in \Sigma(T, (\rho_1, \dots \rho_n))$ by:

$$\begin{aligned}&g \approx g' \text{ if and only if}\\ &\forall C \in \underline{C}, \forall x, y \in T(\rho_1, \dots, \rho_n)(C)\colon\ g(C)(x) = g(C)(y) \Leftrightarrow g'(C)(x)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = g'(C)(y).\end{aligned}$$

Therefore, let us define

$$\Omega = \Omega(\rho_1, \dots, \rho_n) = \Omega(T, (\rho_1, \dots, \rho_n)) \stackrel{\text{def}}{=} \Sigma(T, (\rho_1, \dots, \rho_n))/_{\approx}.$$

The underlying category of the L.F.D., denoted by $\mathcal{F}$, can now be defined by:

$$Ob\mathcal{F} \overset{\text{def}}{=} \coprod_{(\rho_1,\dots,\rho_n)\in\Delta^*_\infty} \Omega(T,(\rho_1,\dots,\rho_n))$$

and an arrow $[(\rho_1,\dots,\rho_n),g/\approx] \to [(\rho'_1,\dots,\rho'_m),g'/\approx]$ in $\mathcal{F}$ can be given by a natural transformation

$$T(\rho_1,\dots,\rho_n)/_{\approx g} \to T(\rho'_1,\dots,\rho'_n)/_{\approx g'}.$$

Furthermore we have a functor D:

$$\begin{array}{rcl} \mathcal{F} & \to & Mod[S] \\ [(\rho_1,\dots,\rho_n),g/\approx] & \mapsto & T(\rho_1,\dots,\rho_n)/_{\approx g} \\ ([(\rho_1,\dots,\rho_n),g/\approx] & \to & [(\rho'_1,\dots,\rho'_n),g'/\approx]) \\ & \mapsto & (T(\rho_1,\dots,\rho_n)/_{\approx g} \to T(\rho'_1,\dots,\rho'_n)/_{\approx g'}). \end{array}$$

We have also a projective cone

$$d\colon \ (d([(\rho_1,\dots,\rho_n),g/\approx])\colon \ T \to T(\rho_1,\dots,\rho_n)/_{\approx g})_{[(\rho_1,\dots,\rho_n),g/\approx]\in Ob\mathcal{F}}$$

where $d([(\rho_1,\dots,\rho_n),g/\approx])$ applies T to the $T/_{\approx g}$ in $T(\rho_1,\dots,\rho_n)/_{\approx g}$.

2.3. THE CONSTRUCTION OF 2.2.2 IS AN L.F.D.

In this subsection we prove the following theorem:

THEOREM 2.3. *The diagram $(\mathcal{F},D,d)$ is locally free for the functor T: $\underline{C} \to Set$.*

In order to prove this theorem we need the following lemma:

LEMMA 2.4. *If $M \in Mod[S]$ is a model of the sketch S, F and G two functors and t: $F \to G$ a natural transformation in $Set^{\underline{C}}$, then for any $\kappa \in \Delta_\infty(G)$ there is a string $\rho \in \Delta_\infty(F)$ and a natural transformation t': $F(\rho) \to G(\kappa)$ whose restriction to F is t. When t is surjective, then t' is surjective too.*

Proof. It follows by the definition of a saturation functor (a saturation functor is a sum of functors) that the lemma is satisfied when we can show this result for $\kappa \in \Delta_0(G)$ and $\rho \in \Delta_0(F)$. So let $\kappa = (\lambda_G, \vartheta_G, \psi_G)$ be an element of

$$\Delta_0(G) = \prod_{I\in\mathbb{I}} \Lambda(\underline{J}^I, G) \times \prod_{I\in\mathbb{I}} \Theta(\underline{J}^I, G) \times \prod_{P\in\mathbb{P}} \Psi(\underline{K}^P, G).$$

– So for any inductive cone $I \in \mathbb{I}$ let λ^I_G be a map from $Ob\underline{J}^I$ to $\mathcal{P}(G^I)$. It is easy to show that for any $J \in \underline{J}^I$ the reciprocal image $t(U^I)^{-1}(\lambda^I_G(J))$ is a subset of F^I. So let us define $\lambda^I_F(J) := t(U^I)^{-1}(\lambda^I_G(J)) \in \mathcal{P}(F^I)$.

– For any inductive cone $I \in \mathbb{I}$ and any zig-zag $J' \overset{\gamma}{\longleftarrow} J \overset{\delta}{\longrightarrow} J''$ of the cone I, $\vartheta_G^I(\gamma,\delta)$ is a subset of $G(U_{J'}^I) \times G(U_{J''}^I)$. It is possible to show that $(t(U_{J'}^I) \times t(U_{J''}^I))^{-1}(\vartheta_G^I(\gamma,\delta)) \subseteq \Theta_F^{\gamma,\delta}(J)$: if we consider $(x_{J'}, x_{J''}) \in (t(U_{J'}^I) \times t(U_{J''}^I))^{-1}\vartheta_G^I(\gamma,\delta))$, then for any $x_J \in F(U_J^I)$ we must have $F(\gamma)(x_J) \neq x_{J'}$ or $F(\delta)(x_J) \neq x_{J''}$, because if not, then it would follow that $G(\gamma)(t(U_J^I)(x_J)) = t(U_{J'}^I)(x_{J'})$ and $G(\delta)(t(U_J^I)(x_J)) = t(U_{J''}^I)(x_{J''})$. This would be a contradiction to $(x_{J'}, x_{J''}) \in (t(U_{J'}^I) \times t(U_{J''}^I))^{-1}\vartheta_G^I(\gamma,\delta))$. Therefore we have shown that $(t(U_{J'}^I) \times t(U_{J''}^I))^{-1}(\vartheta_G^I(\gamma,\delta)) \subseteq \Theta_F^{\gamma,\delta}(J)$ and so we can define $\vartheta_F^I(J) := \coprod_{\gamma,\delta}(t(U_{J'}^I) \times t(U_{J''}^I))^{-1}(\vartheta_G^I(\gamma,\delta))$ with $\vartheta_F^I \in \Theta(\underline{J}^I, F)$.
– For any projective cone $P \in \mathbb{P}$, $\psi_G^P \subseteq \prod_{K \in \underline{K}^P} G(V_K^P)$. So let us define $\psi_F^P := (\prod_{K \in \underline{K}^P} t(V_K^P))^{-1}(\psi_G^P)$ and therefore we have $\psi_F^P \in \Psi(\underline{K}^P, F)$.

So we have defined for a $\kappa = (\lambda_G, \vartheta_G, \psi_G) \in \Delta_0(G)$ a triple $t^*(\kappa) = \rho = (\lambda_F, \vartheta_F, \psi_F) \in \Delta_0(F)$ and by the way how $\rho = (\lambda_F, \vartheta_F, \psi_F)$ is established, it follows that there is a t': $F(\rho) \to G(\kappa)$ whose restriction is t. For instance, for $x \in \lambda_F^I(J)$ we can define $t'(W)(x) = t(U^I)(x)$. Furthermore it is obvious that t' is surjective when t is surjective. □

Proof of Theorem 2.3. According to Definition 1.2.1 we have to verify for any model $M \in Mod[S]$ and any morphism h: $T \to M$ in $Set^{\underline{C}}$ two properties:

– First, we must find $[(\rho_1, \ldots, \rho_n), g/_{\approx}] \in Ob\mathcal{F}$ and a factorization of $(T \overset{h}{\longrightarrow} M)$ by $T \to T(\rho_1, \ldots, \rho_n)/_{\approx g}$. In ([3, pp. 32–41] or [4]) a ρ: $\mathbf{N} \to \coprod_{n \in \mathbf{N}} \Delta_n$ and a morphism g: $T(\rho) \to M$ are constructed such that $T(\rho)/_{\approx g}$ is a model and such that $T(\rho)/_{\approx g}$ factorizes $(T \overset{h}{\longrightarrow} M)$.
– Furthermore, if the two strings of the L.F.D. $([(\rho_1, \ldots, \rho_n), g/_{\approx}], m)$ and $([(\rho_1', \ldots, \rho_n'), g'/_{\approx}], m')$ factorize the morphism $T \overset{h}{\longrightarrow} M$, then we should find a factorizing zig-zag between $([(\rho_1, \ldots, \rho_n), g/_{\approx}], m)$ and $([(\rho_1', \ldots, \rho_n'), g'/_{\approx}], m')$.

Let us write $\bar{\rho} = (\rho_1, \ldots, \rho_n)$ and $\bar{\rho}' = (\rho_1', \ldots, \rho_n')$. Then let $[\bar{\rho}, g/_{\approx}]$ and $[\bar{\rho}', g'/_{\approx}]$ satisfy the first property. Without restriction to generality we can suppose that $n = m$ (if this is not the case, we can prolong the shortest series by empty saturations). At first, let us consider the pushout $Q := (T(\bar{\rho})/_{\approx g} +_T T(\bar{\rho})/_{\approx g'})$ which is generated by the morphisms $T \to T(\bar{\rho})/_{\approx g}$ and $T \to T(\bar{\rho}')/_{\approx g'}$. The universal property of this pushout delivers a unique morphism q: $Q \to M$ such that q commutes with m and m'.

Applying ([3, pp. 32–41] or [4]) to the functor Q, we can find a map μ: $\mathbf{N} \to \coprod_{n \in \mathbf{N}} \Delta_n(Q)$ and a morphism t: $Q(\mu) \to M$ such that $Q(\mu)/_{\approx t}$ is a model and such that the morphism q factorizes by $Q(\mu)/_{\approx t}$. Until the end of this proof we show that there is a $(\bar{\xi}, e/_{\approx}) = [(\xi_1, \ldots, \xi_{n+1}), e/_{\approx}] \in Ob\mathcal{F}$ such that $T(\bar{\xi})/_{\approx e}$ is isomorphic to $Q(\mu)/_{\approx t}$, and that there are arrows $(\bar{\rho}, g/_{\approx}) \to (\bar{\xi}, e/_{\approx})$ and $(\bar{\rho}', g'/_{\approx}) \to (\bar{\xi}, e/_{\approx})$ which deliver the desired zig-zag. □

To achieve this let us prove the following lemma:

LEMMA 2.5. *With the above notations, there is a string*

$$\bar{\xi} = (\xi_1, \dots, \xi_n) \in \Delta^*_\infty(T)$$

such that $T(\bar{\xi}) = T(\bar{\rho}) +_T T(\bar{\rho}') + R(\bar{\rho}, \bar{\rho}')$ *where* $R(\bar{\rho}, \bar{\rho}')$ *is a sum of subfunctors* (R *means "rest") of one of the* $T(\xi_1, \dots, \xi_i)$ *with* $i \in \{1, \dots, n\}$. *Furthermore there are natural transformations*

$$T(\xi_1, \dots, \xi_i) \to Q$$

($i \in \{1, \dots, n\}$) *such that the corresponding diagram over* M *is commutative and such that* $T(\xi_1, \dots, \xi_n) \to Q$ *is surjective.*

Proof. The morphisms $T(\bar{\rho}) \to Q$ and $T(\bar{\rho}') \to Q$ induce morphisms $T(\rho_1, \dots, \rho_i) \to Q$ and $T(\rho'_1, \dots, \rho'_i) \to Q$ for $i \in \{1, \dots, n\}$. Let us construct successively functors $T(\xi_1, \dots, \xi_i)$ and natural transformations $T(\xi_1, \dots, \xi_i) \to Q$ for $i \in \{1, \dots, n\}$. For ρ_1 and $\rho'_1 \in \Delta_\infty(T)$ let us define ξ_1: $\mathbb{N} \to \coprod_{n \in \mathbb{N}} \Delta_n(T)$ such that $\xi_1(n) := \rho_1(l)$ for $n = 2l$ and $\xi_1(n) := \rho'_1(l)$ for $n = 2l + 1$. It is obvious that $T(\rho_1) +_T T(\rho'_1) = T(\xi_1)$. The universal property of this pushout delivers a canonical morphism $T(\xi_1) \to Q$. We want to show in a first step that ρ_2 et $\rho'_2 \in \Delta_\infty(T(\xi_1))$. In order to prove this, we shall show that $\Delta_0(T(\rho_1)) \subseteq \Delta_0(T(\xi_1))$ (and $\Delta_0(T(\rho'_1)) \subseteq \Delta_0(T(\xi_1))$). Therefore let us show that $\Lambda(\underline{J}^I, T(\rho_1)^I) \subseteq \Lambda(\underline{J}^I, T(\xi_1)^I)$, $\Theta(\underline{J}^I, T(\rho_1)) \subseteq \Theta(\underline{J}^I, T(\xi_1))$ and $\Psi(\underline{K}^P, T(\rho_1)) \subseteq \Psi(\underline{K}^P, T(\xi_1))$ for $I \in \mathbb{I}$ and $P \in \mathbb{P}$.

- The first inclusion is satisfied if we can show that $T(\rho_1)^I \subseteq T(\xi_1)^I$. It is possible to write the functors $T(\rho_1)$ and $T(\xi_1)$ by separating T from a "rest"-functor in a canonical way: $T(\rho_1) = T + R(\rho_1)$ and $T(\xi_1) = T + R(\rho_1) + R(\rho'_1)$. It follows that $T(\rho_1)^I = T^I + R(\rho_1)^I$ and $T(\xi_1)^I = T^I + R(\rho_1)^I + R(\rho'_1)^I$. So we can see that $T(\rho_1)^I \subseteq T(\xi)^I$ and thus we obtain the first inclusion.
- With regard to the second inclusion let us prove $\Theta^{\gamma,\delta}_{T(\rho_1)}(J) \subseteq \Theta^{\gamma,\delta}_{T(\xi_1)}(J)$. Let $(x_{J'}, x_{J''}) \in \Theta^{\gamma,\delta}_{T(\rho_1)}(J)$ and $x_J \in T(\xi_1)(U^I_J)$. Suppose that $T(\xi_1)(\gamma)(x_J) = x_{J'}$ and $T(\xi_1)(\delta)(x_J) = x_{J''}$. As the different sum-strata are completely separated, it follows that $x_J \in T(\rho_1)(U^I_J)$ and $(x_{J'}, x_{J''}) \notin \Theta^{\gamma,\delta}_{T(\rho_1)}(J)$ which is a contradiction. So $\Theta^{\gamma,\delta}_{T(\rho_1)}(J) \subseteq \Theta^{\gamma,\delta}_{T(\xi_1)}(J)$. It follows that $\Theta^I_{T(\rho_1)}(J) \subseteq \Theta^I_{T(\xi_1)}(J)$ and finally that $\Theta(\underline{J}^I, T(\rho_1)) \subseteq \Theta(\underline{J}^I, T(\xi_1))$.
- The third inclusion is an immediate consequence of the following inclusions $T(\rho_1)(V^P_K) \subseteq T(\xi_1)(V^P_K)$.

Thus we get $\Delta_0(T(\rho_1)) \subseteq \Delta_0(T(\xi_1))$, $\Delta_0(T(\rho'_1)) \subseteq \Delta_0(T(\xi_1))$. By induction we also have that $\Delta_k(T(\rho_1)) \subseteq \Delta_k(T(\xi_1))$, $\Delta_k(T(\rho'_1)) \subseteq \Delta_k(T(\xi_1))$. It follows that $\Delta_\infty(T(\rho_1)) \subseteq \Delta_\infty(T(\xi_1))$, $\Delta_\infty(T(\rho'_1)) \subseteq \Delta_\infty(T(\xi_1))$, and therefore ρ_2

and $\rho_2' \in \Delta_\infty(T(\xi_1))$. We can write again $T(\xi_1) = T(\rho_1) + R(\rho_1')$ and obtain by the application of the definition of a saturation functor:

$$T(\xi_1)(\rho_2) = T(\rho_1) + R(\rho_1') + \coprod_{k\in\mathbb{N}} (T(\rho_1) + R(\rho_1'))_{l(\rho_2(k))+1}(\rho_2(k)).$$

By considering this definition (see Section 2.2.1) we can separate the different elements in the following way:

$$T(\xi_1)(\rho_2) = T(\rho_1)(\rho_2) + R(\rho_1') + \mathbb{N}\times R(\rho_1') = T(\rho_1)(\rho_2) + \mathbb{N}\times R(\rho_1').$$

For $T(\xi_1)(\rho_2')$ we have a similar result:

$$T(\xi_1)(\rho_2') = T(\rho_1)(\rho_2) + \mathbb{N}\times R(\rho_1).$$

Since $R(\rho_1)$ and $R(\rho_1')$ are subfunctors of $T(\xi)$, there are two canonical morphisms $T(\xi_1)(\rho_2) \to Q$, $T(\xi_1)(\rho_2') \to Q$. Next we can define ξ_2: $\mathbb{N} \to \coprod_{n\in\mathbb{N}} \Delta_n(T(\xi_1))$ by ρ_2 and ρ_2' in the same way as ξ_1. It is obvious that $T(\xi_1)(\rho_2) +_{T(\xi_1)} T(\xi_1)(\rho_2') = T(\xi_1)(\xi_2)$. The universal property of this pushout delivers a unique morphism $T(\xi_1)(\xi_2) \to$. By calculating this pushout we can express $T(\xi_1)(\xi_2)$ by $T(\xi_1)(\xi_2) = T(\rho_1)(\rho_2) +_T T(\rho_1')(\rho_2') + \mathbb{N}\times R(\rho_1) + \mathbb{N}\times R(\rho_1')$. There are two possibilities of "pulling" the functor T: $T(\xi_1)(\xi_2) = T(\rho_1)(\rho_2) + R(\rho_1', \rho_2') + \mathbb{N}\times R(\rho_1) + \mathbb{N}\times R(\rho_1')$ and $T(\xi_1)(\xi_2) = T(\rho_1')(\rho_2') + R(\rho_1, \rho_2) + \mathbb{N}\times R(\rho_1) + \mathbb{N}\times R(\rho_1')$ where $R(\rho_1', \rho_2')$ and $R(\rho_1, \rho_2)$ are two "rest"-functors depending on (ρ_1', ρ_2') and (ρ_1, ρ_2). We can actually express $T(\rho_1)(\rho_2)$ even more simplier: By $T(\xi_1)(\xi_2) = T(\rho_1)(\rho_2) + R(\rho_1, \rho_1', \rho_2')$ or by $T(\xi_1)(\xi_2) = T(\rho_1')(\rho_2') + R(\rho_1', \rho_1, \rho_2)$ with two "rest"-functors $R(\rho_1, \rho_1', \rho_2')$ and $R(\rho_1', \rho_1, \rho_2)$. We can continue now on this way by induction and so we obtain for $i \in \{1, \dots, n\}$ saturation functors $T(\xi_1, \dots, \xi_i)$ such that $T(\bar{\xi}) = T(\bar{\rho}) +_T T(\bar{\rho}') + R(\bar{\rho}, \bar{\rho}')$ where $R(\bar{\rho}, \bar{\rho}')$ is a sum of subfunctors of one of the $T(\xi_1, \dots, \xi_i)$ with $i \in \{1, \dots, n\}$. Furthermore we have a unique morphism $T(\bar{\xi}) \to Q$ which is surjective. We can define morphisms $T(\xi_1, \dots, \xi_i) \to M$ by combining $T(\xi_1, \dots, \xi_i) \to Q$ with q: $Q \to M$. So we obtain a commutative diagram over M. □

We can apply now Lemma 2.4 to the string $\mu \in \Delta_\infty(Q)$ and find a string $\xi_{n+1} \in \Delta_\infty(T(\bar{\xi}))$ and a surjective natural transformation $T(\bar{\xi})(\xi_{n+1})$. By combining this natural transformation with $Q(\mu) \to Q(\mu)/_{\approx t}$ we obtain a surjective natural transformation e: $T(\bar{\xi})(\xi_{n+1}) \to Q(\mu)/_{\approx t}$. Therefore we have a model $T(\bar{\xi})(\xi_{n+1})/_{\approx e}$ which is generated by a point of the L.F.D. and which is isomorphic to $Q(\mu)/_{\approx t}$. The arrows $(\bar{\rho}, g/_{\approx}) \to (\bar{\xi}, e/_{\approx})$ and $(\bar{\rho}', g'/_{\approx}) \to (\bar{\xi}, e/_{\approx})$ deliver the desired zig-zag.

2.4. Refinement: The construction of 2.2.4 is relatively filtered

We can even show the result

THEOREM 2.4. *The locally free diagram* $(\mathcal{F}, D, d)$ *is relatively filtered for the functor* T: $\underline{C} \to Set$.

Proof. According to Definition 1.2.4 we have to verify for any model $M \in Mod[S]$ and any morphism h: $T \to M$ in $Set^{\underline{C}}$ two properties:

– The first point of the "filtered"-property has indeed been shown in the second step of the proof of 2.3.
– Secondly, we should show that for $([\bar{\rho}, g/_{\approx}], m)$ and $([\bar{\rho}', g'/_{\approx}], m')$ factorizing the morphism $T \xrightarrow{h} M$ and for u, v: $[\bar{\rho}, g/_{\approx}] \to [\bar{\rho}', g'/_{\approx}]$, there is a $[\bar{\xi}, e/_{\approx}]$ in $Ob\mathcal{F}$ and an arrow w: $[\bar{\rho}', g'/_{\approx}] \to [\bar{\xi}, e/_{\approx}]$ in $Arr\mathcal{F}$ with $w \circ u = w \circ v$ providing the corresponding commutative diagram on the models (see the Definition 1.2.4). We construct the proof in a similar way to the second point of the proof of Theorem 2.3.

The universal property of the coequaliser $Q := Coeq(D(u), D(v))$ will deliver a unique morphism q: $Q \to M$ commuting with $m, m', D(u)$ and $D(v)$. Applying ([3, pp. 32–41] or [4]) to the functor Q, we can find a map μ: $\mathbf{N} \to \coprod_{n \in \mathbf{N}} \Delta_n(Q)$ and a morphism t: $Q(\mu) \to M$ such that $Q(\mu)/_{\approx t}$ is a model and such that the morphism m factorizes by $Q(\mu)/_{\approx t}$. Up to the end of this proof we shall show that there is a $(\bar{\xi}, e/_{\approx}) \in Ob\mathcal{F}$ such that $T(\bar{\xi})/_{\approx e}$ is isomorphic to $Q(\mu)/_{\approx t}$, and that there is an arrow $(\bar{\rho}', g'/_{\approx}) \to (\bar{\xi}, e/_{\approx})$ which delivers the desired morphism.

It is easy to see that the morphism $T(\bar{\xi}) \to T(\bar{\rho}')/_{\approx g'} \to Q$ is surjective. Now we have come to a point which has already developped in the second part of the proof of Theorem 2.3. Therefore, we can complete the proof in a completely analogous way to the proof of Theorem 2.3. □

3. Conclusive Remark

By our construction, we hope to get the first step in establishing a general representation theory for Locally Free Diagrams. One of the aims of a representation theory is to represent the elements of a set with an algebraic structure S_0 by the global continuous sections of a sheaf whose fibers have an algebraic structure S_1 which is richer and more effective than S_0 (see [2]). By doing this we obtain the properties of S_0 by those of S_1. In the frame of representation theory, Diers ([2]) has established a spectral theory for the so-called multi-free structures. Our common research project is to apply these spectral notions to more general structures, that is to say in the case when the Locally Free Diagrams exist.

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