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THE THEORY OF SKETCHES (REVISITED)

by

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This talk is a survey of recent results obtained by myself and Christian Lair. The proofs are detailed in our papers ([4]), ([5]) and ([6]).

We have been spurred on this subject by two false conceptions of some categoricians :

(C_1) : " The calculus with a sketch S is too much subtle, and it is better to work with the type (i.e. the completion) of S "

(C_2) : " The calculus with a mixed sketch is too much general and cannot produce results as for instance those produced by the calculus with sites or the calculus with localizable categories "

The falseness of C_2 is not so immediate to detect, but follows from our results hereunder ; these results work also against the conception C_1, but concerning C_1 it is also interesting to insist on its philosophical connotation :

The power of Category Theory (C.T.) consists in two complementary things : on the first hand C.T. provides a large background (<u>the philosophy of adjunctions</u>) for the quest of "good concepts" ; on the second hand C.T. provides a precise tool (<u>the diagrammatical method</u>) for the effective attack of "essentially combinatorial problems" (as for instance the study of limits in a category of models).

In fact C_1 lay on the forgetting of the second hand above-mentioned.

It is as much false as are false its classical analogues (e.g. : " the calculus with generators of a group is too much subtle, and it is better to work with all the group ", etc.).

With the Theory of Sketches (S.T.) we are not looking for a kind of "aesthetic or conceptual satisfaction", this is not the point. The fact is that S.T. is an analytical model theory (in the same way one can speak of the analytical geometry), and the question is just : is it a simple and successful tool ? So, in order to be clear on this question, I will not speak here of different possible extensions of the calculus of sketches initiated in (4), (5) or (6) (e.g. sketches over a given sketch and theories in a given theory, enriched sketches, calculus with internal formulas and internal trees, links with the calculus of exact squares and the deduction in a fibered category . For this last point cf. also my talk at the Amiens'80 meeting : " Qu'est-ce que la logique dans une catégorie ? ",to appear in Cahiers Top. Géo. Dif.).

1. Sketches and sketchable categories.

Definition 1 (Ehresmann, in (3)). An abstract sketch - or shortly a sketch- is a data $S = (\underline{S}, \underline{P}, \underline{Y})$ where $\underline{S}$ is a category, $\underline{P}$ is a family of distinguished projective cones on $\underline{S}$, and $\underline{Y}$ is a family of distinguished inductive cones on $\underline{S}$. A realization R of S is a functor $R : \underline{S} \longrightarrow \underline{SET}$ continuous and co-continuous (i.e. for all $p \in \underline{P}$ the cone Rp in $\underline{SET}$ is a projective limit cone, and for all $y \in \underline{Y}$ the cone Ry in $\underline{SET}$ is an inductive limit cone).

Definition 2 (Guitart-Lair, in (4)). A concrete sketch is a data $S = (\underline{S}, \underline{F})$ where $\underline{S}$ is a graph (i.e. an object of the category $\underline{SET}^{(\cdot \rightrightarrows \cdot)}$) and $\underline{F}$ a family of distinguished projective cones on $\underline{SET}^{\underline{S}}$. An element $f = (f_I : V \longrightarrow C_I)_{I \in \underline{I}}$ of $\underline{F}$ is a $\underline{S}$-formula, V is its "variable", and the C_I are the "conditions". A realization R of S is a functor $R : \underline{S} \longrightarrow \underline{SET}$ such that for all f in $\underline{F}$ we have

$$\underrightarrow{\mathrm{Lim}}\,\mathrm{Hom}(C_I, R) \xrightarrow[\simeq]{\text{iso}} \mathrm{Hom}(V, R).$$

Definition 3 (Andreka-Nemeti, in (1)). An axiomatization is a data $S = (\underline{S}, \underline{A})$

where $\underline{S}$ is a category and $\underline{A}$ a family of distinguished discrete projective cones on $\underline{SET}^{\underline{S}}$.

A realization R of S is a functor $R : \underline{S} \longrightarrow \underline{SET}$ such that for all element $a = (a_i : D \longrightarrow B_i)_{i \in I}$ in $\underline{A}$ we have

$$\coprod_{i \in I} \mathrm{Hom}(B_i, R) \xrightarrow{\text{epi}} \mathrm{Hom}(D, R).$$

Let S be an abstract sketch, a concrete sketch, or an axiomatization. The category of all natural transformations between realizations of S is denoted by SET^S, and the class of its objects is denoted by $(SET^S)_o$.

Theorem 1 (from chp.2 and chp.4 of (4)). The following conditions on a category $\underline{X}$ are equivalent :

(i) $\underline{X}$ is naturally sketchable in the sense of (3) i.e. $\underline{X}$ is equivalent to a category SET^S with S an abstract sketch.

(j) $\underline{X}$ is equivalent to a category SET^S with S a concrete sketch.

(k) $\underline{X}$ is equivalent to a category SET^S with S an axiomatization.

So, from now on, the distinction between (i) and (j) or between (i) and (k) is mainly of a psychological interest, in the same vein than the distinction between abstract manifolds and embedded manifolds.

About sketchability (in the sense (i)) we can also recall a theorem of Lair (published in 1971) which says that $\underline{X}$ is naturally sketchable iff $\underline{X}$ is naturally sketched by $S_{\underline{X}} = (\underline{SET}^{\underline{X}}, \underleftarrow{\mathrm{Lim}}, \underrightarrow{\mathrm{Lim}})$ where $\underleftarrow{\mathrm{Lim}}$ (resp. $\underrightarrow{\mathrm{Lim}}$) is the class of all projective limit cones (resp. inductive limit cones) on $SET^{\underline{X}}$.

Now, let S and S' be two abstract sketches. A morphism from S to S' is a functor $F : \underline{S} \longrightarrow \underline{S}'$ such that $F(\underline{P}) \subset \underline{P}'$ and $F(\underline{Y}) \subset \underline{Y}'$. We denote by $\underline{SKE}$ the 2-category of all natural transformations between these morphisms.

Definition 4 (Guitart-Lair, in (5) and (6)). A category $\underline{X}$ is sketchable if there is a category $\mathcal{E}$ internal in $\underline{SKE}^{op}$ such that $\underline{X}$ is equivalent to $(SET^{\mathcal{E}})_o$. Roughly speaking $\underline{X}$ is sketchable if its objects (resp. its morphisms) can be described as functors $\underline{S}_o \longrightarrow \underline{SET}$ (resp. $\underline{S}_1 \longrightarrow \underline{SET}$) satisfying some continuity and co-continuity conditions. And $\underline{X}$ is naturally sketchable in the case where $S_1 = S_o \otimes \mathbf{2}$.

Theorem 2(from (5)). In general a sketchable category is not naturally sketchable. If $\underline{X}$ is sketchable by $\underline{\mathcal{S}} = (S_o \overset{i_o}{\underset{i_1}{\rightrightarrows}} S_1 \longrightarrow S_2)$, if S_o is purely projective (i.e. without distinguished inductive cones), and if there is a functor $V : \underline{S_1} \longrightarrow \underline{S_o} \times 2$ such that $V.i_o = d_o$ and $V.i_1 = d_1$, then there is a naturally sketchable category $\underline{B}$ - namely $\underline{B} = SET^{S_o}$ -, and a co-monad $\mathbb{G}$ on $\underline{B}$ such that $\underline{X} = \text{Kleisli}(\mathbb{G})$.

In (5) there are others criterions. This type of results allows to study by the analytical method of sketches various categories of "lax" morphisms.

2. Examples.

Of course the algebraical theories of Lawvere, the algebraical theories of Bénabou, the categories of algebras of monads on SET, the categories of sheaves on sites, are examples of purely projective sketches.

A first interesting example of a mixed sketch has been furnished by Burroni in 1970, for the category TOP of topological spaces.

Theorem 3 (from (6)). If $\underline{X}$ is split cofibered over SET, then $\underline{X}$ is sketchable.

Theorem 4 (from (6)). If $\underline{X}$ is a category mod T of morphisms between models of an arbitrary finitary first order theory, then $\underline{X}$ is sketchable.

Theorem 5 (from (4)). If $\underline{X}$ is the category of models in SET of a site, then $\underline{X}$ is naturally sketchable.

It will be interesting to characterize the "shape" of the sketches corresponding to sites. We havn't got time to do it. But we have completely solved the question for localizable categories :

Theorem 6 (from (4)). A category $\underline{X}$ is localizable in the sense of Diers (2) iff $\underline{X}$ is naturally sketchable by a sketch S where the distinguished inductive cones have discrete bases (i.e. they represent some sums).

N.B. In fact the proof of this theorem 6 use of the theorem 12 (§ 4).

The class of categories of models of sites and the class of localizable categories are not "stable by negation" (so, the category of rings which are not fields is not localizable). By contrast we have :

Theorem 7 (from [6]). Any boolean (finite or not) combination of sketches is again a sketch (a mixed sketch in general, even if the datas are purely projective or purely inductive).

Theorem 8 (from [6]). If X is sketchable by a purely inductive sketch, then X is sketchable by a purely projective sketch (but possibly this projective sketch will be large).

So, up to size conditions, purely inductive and purely projective sketches are of the same nature, essentially algebraical. Consequently the full power of the Theory of Sketches is situated in the mixed case.

5. *Limits of models.*

One of the applications of projective sketches has been the classification of associative unitary tensors in algebraical categories over SET (after 1972, in the works of A. & C. Ehresmann, Foltz-Lair, Foltz-Kelly-Lair). Concerning tensors I have shown their existence in a category of "commutative algebras" K^T over a monoidal category K. I worked with the point of view of monads, and strictly speaking my result does not concern sketches (the base category K is not SET, and even is not necessarily sketchable). Nevertheless it is well known that we have to thing of the Kleisli category of a monad as a sketch for its algebras ; so in this kind of results the part played by sketches is to be clarified.

The question of tensors involved a special case of the essential problem of the construction of inductive limits in algebras. And if X is a non-algebraical category of models, X is sketched, but not necessarily by a purely projective sketch, so that the existence of projective limits in X makes problem.

Concerning this question of limits in models the fundamental basic idea is that *the main obstruction* for the computation of limits in a naturally sketchable category *is concentrated in the problem of commutation of limits in SET*.

In [4] we give some criterions for the computations of projective limits, inductive limits, partial limits, local limits, ultraproducts. In this talk I will give only two examples.

Theorem 9 (from [4]). Let $\underline{X}$ be a naturally sketchable category, $\underline{X} = \underline{SET}^S$ with $S = (\underline{S}, \underline{P}, \underline{Y})$.
We denote by Base($\underline{Y}$) the set of categories which are bases of cones of $\underline{Y}$, and we denote by Com Base($\underline{Y}$) the set of categories $\underline{C}$ such that for all $\underline{B} \in$ Base($\underline{Y}$) the projective limits in $\underline{SET}$ indexed by $\underline{C}$ commute with the inductive limits in $\underline{SET}$ indexed by $\underline{B}$. Then we have :
The category $\underline{X}$ have all projective limits indexed by any category $\underline{C} \in$ Com Base($\underline{Y}$).
The dual result holds for inductive limits.

This theorem 9 is very easy, and as a corollary we have that a localizable category admits connected projective limits, because in $\underline{SET}$ these limits commute with sums, and because of theorem 6. So, just looking to the shape of the sketch of a field, it is trivial that the category of fields admits pullbacks.

But why the category of fields does not admit products ? In fact it is really because the product in the category of rings of two fields is not a field, and because the sketch of field is limited in the following sense :

Definition 5 (from [4]). Let $S = (\underline{S}, \underline{P}, \underline{Y})$ be a sketch, and let $S_{proj} = (\underline{S}, \underline{P}, \emptyset)$ be its " projective part ". S is said to be limited if all object V of $\underline{S}$ is the top of a projective cone $p \in \underline{P}$, $p = (p_I : V \longrightarrow P_I)_{I \in \underline{I}}$ such that for all object I of $\underline{I}$ the functor $\mathrm{Hom}(P_I, -) : \underline{S} \longrightarrow \underline{SET}$ is a realization of S.

Theorem 10 (from [4]). If a sketch S is limited then the inclusion $SET^S \longrightarrow SET^{S_{proj}}$ commutes with all small projective limits.

4. Toward the spectrum : the small locally free diagram.

In the case of purely projective sketches an essential tool was the associated sheaf theorem, and a mixed version of it was to find. We have it (theorem 11 below).

Definition 6 (from [4]). Let c be a cardinal. A c-injective sketch is a sketch $S = (\underline{S}, \underline{P}, \underline{Y})$ such that for all $y = (y_J : B_J \longrightarrow V)_{J \in \underline{J}}$ element of $\underline{Y}$, for all realization $R : S \longrightarrow SET$, and for all $x \in R(V)$ we have

$$\mathrm{Card}\ R(y_J)^{-1}(x) \leqslant c .$$

Theorem 11 (from (4)). If S is a small c-injective sketch, then each realization $R : S_{proj} \longrightarrow SET$ (resp. each functor $R : \underline{S} \longrightarrow \underline{SET}$) have a small locally free diagram (D, d) in SET^S. This means that $D : \underline{A} \longrightarrow SET^S$ is a functor with domain a small category $\underline{A}$, $d = (d_A : R \longrightarrow D_A)_{A \in \underline{A}}$ is a projective cone in $\underline{SET}^{\underline{S}}$ with base D and with top R, and these datas induce, for all G in SET^S (G realization of the mixed sketch S), an isomorphism

$$\underset{A \in \underline{A}^{op}}{\underrightarrow{\mathrm{Lim}}} \mathrm{Hom}_{SET^S}(D_A, G) \xrightarrow{\sim} \mathrm{Hom}_{SET^{S_{proj}}}(R, G).$$

Theorem 12 (from (4)). If S is a small sketch where the distinguished inductive cones have discrete bases, then each realization $R : S_{proj} \longrightarrow SET$ has a small locally free family $((D_m)_{m \in M}, (d_m)_{m \in M})$ in SET^S. This means that M is a set, $(D_m)_{m \in M}$ is a M-family of realization of S, $(d_m)_{m \in M}$ a M-family of morphisms $d_m : R \longrightarrow D_m$ in $SET^{S_{proj}}$, and these datas induce, for all G in SET^S an isomorphism

$$\coprod_{m \in M} \mathrm{Hom}_{SET^S}(D_m, G) \xrightarrow{\sim} \mathrm{Hom}_{SET^{S_{proj}}}(R, G).$$

In the "Diers version" of the spectrum there is, as an hypothesis, the existence of locally free families : our theorem 12 says that this hypothesis is always satisfied in the case of localizable categories and localizable functors.

In order to understand in the same way the beginning of the "topos version" of the spectrum w.r.t. a geometrical theory (Coste, Makkaï-Reyes),we have to note that the proof of Theorem 5 consist in the use of Theorem 1 for the axiomatization $(\underline{C}, YON(G_o(\underline{C}))$ where $YON : \underline{C} \rightarrow (\underline{SET}^{\underline{C}})^{op}$ and $(\underline{C}, G_o(\underline{C}))$ a site given by a basic system $G_o(C)$ of covering families . And then we have :

Theorem 13 (from (4)). Let $(\underline{S}, \underline{A})$ an axiomatization (e.g. $(\underline{C}, YON(G_o(\underline{C})))$). If for all $a = (a_i : D \longrightarrow B_i)_{i \in \underline{I}}$ in $\underline{A}$ we have that each a_i is an epimorphism, then the sketch associated (by Theorem 1, (i)$\longleftrightarrow$(k)) is 1-injective, and the conclusion of theorem 11 is true.

Now to go to a spectral analysis of an arbitrary first order theory, the key is theorem 11 : the deeper step in the description of the spectrum is the construction of small locally free diagrams (s.l.f.d.) as in theorem 11, and then the "total spectrum" of R will be the description of all connections between the various s.l.f.d. on R. This will be exposed in details elsewhere.

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