EQUATIONAL TRANSLATION OF SET THEORITICAL NOTIONS (+)

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I. INVOLUTIVE MONADS.- The main problem is to find an "equational" context which exists in *Sets* and which would allows us to develop in an equational way the theory of the definition (of the types of structures).

Roughly speaking we have to solve the equation

" topos = finitely complete cat. + cartesian closed cat. + ? "

Let us begin with an abelian sup-monoid $A = (\underline{A}, \sup, k)$, that is to say a complete lattice $(\underline{A}, \sup)$ and an abelian monoid $(\underline{A}, k)$ (whose unit is denoted by e) where the law is a sup-lattice morphism (examples of this situation are distributive lattices, and, also, the set $[0, 1]$ with its usual order and multiplication)

For a set X let A^X be denoted by FX, and let $i_X : X \longrightarrow FX$ and $d_X : FX \longrightarrow F^2X$ be defined by

$$i_X xx' = \begin{cases} e \text{ if } x = x' \\ 0 \text{ if } x \neq x' \end{cases} \qquad \text{and} \qquad d_X pp' = \sup_{x \in X} k(px, p'x).$$

The system (F, i, d) is a *"contravariant standard construction"*(c.s.c.) over *Sets*, i.e. satisfies the 4 equations given in ([1]) or ([2]). It is a theorem that over each category C there is a bijection between the set of c.s.c. over C and the set of *involutive monads* over C (we call involutive monad (i.m.) over C a pair (P, I) where P is a monad over C and I an involution on the Kleisli's category *Kl*P of P).

A definition of a complemented i.m.$ (c.i.m.) is given in ([2]), and an equivalent one can be found in ([0]). In my talk at the Open House on Category Theory organised at the University of Sussex (July 74) I have given a lot of examples (on a topos, on the category of relations, on the "special" category of compact spaces, on the category of "quasi-topologies", etc...). As many people here were last week in Sussex, I would not repeat this list. Looking for some examples in *Cat* I have studied the notion of a machine (cf. my lecture at the Amiens'Meeting on July 73) : actually this is studied now by R. Street (and exposed in his lecture here) in the context of 2-categories, which suggests how to extend the concept of involutive monad into the 2 dimensional case. Let us notice that, in *Cat*, the case $F = Sets^{(-)^*}$ is not an example of an involutive monad only because of the question of size. However, the case $F = 2^{(-)^*}$ is an example of i.m. in *Cat*.

(+) The summary of a conference at Oberwolfach (August 74).

We denote by U_A the i.m. over *Sets* exhibited at the beginning of this paragraph. In the category of i.m. over *Sets* we have the simplicial object $\hat{U}$

$$\hat{U} : \quad U_1 \xleftarrow{\ \varepsilon\ } U_2 \underset{\beta}{\overset{\alpha}{\longleftarrow}} \!\!\xrightarrow{\ u\ } U_3 \xleftarrow{\ \kappa\ } U_4 \longleftarrow \cdots$$

which in fact is a category, and comes from the involutive monad $2^{(-)}$ over *Cat*.

Everyone knows how to make use of U_1 and U_2 to work out some mathematical notions. But the question is : what comes after 1 and 2 ? A somewhat natural reaction would then make us think about using U_3 and U_4.

Notice that if we add to U_N the data of all the maps from U_N to itself coming from the maps ε, α, β, u, κ, ... in the simplicial object $\hat{U}$, we obtain a system reacher than an i.m., which we denote by $\vec{U}_N$.

II. STRUCTURAL EQUATIONS.- Let C be a category and (F, i, d) = U an i.m. over C. An *equation* for $f_1, \ldots, f_n$ in U consists of an identity " A = B " where A and B are composites in C of morphisms of the form $F^m f_i$, $F^m i_F p_{f_i}$ or $F^m d_F p_{f_i}$. Clearly for every abelian sup-monoid A, an equation E can be interpreted as a formula written $J_A E(f_1, \ldots, f_n)$. Hence, every such equation defines a theory $T_A E$ whose models are (by definition) n-tuples $(f_1, \ldots, f_n)$ of maps verifying the identity "$A(f_1, \ldots, f_n) = B(f_1, \ldots, f_n)$" in U_A.

Main problem : If T is a type of structure, could we find an equation $E_A T(f_1, \ldots, f_n)$ whose solutions in U_A are exactly the models of T ?

Such an equation, if it exists, will be called a "*structural equation of the theory* T *in the context* U_A" (or simply a s.e. of T in A).

Nota.- If A and A' are two abelian sup-monoids then, given a theory T, we can transform it into a new theory T' by "modifying the underlying logics" as follows : if T admits a structural equation $E_A T$ in A, then the interpretation $J_{A'}(E_A T)$ defines the new theory T'.

The following theories admit structural equations for A = 2 : the theory of the void set, the theory of the set 2^n, theories of relations, of order relations, of congruences, of injections, of surjections, of complete atomic boolean algebras, of points (elements).

When we work out the notion of a structural equation in the context of c.i.m., we get structural equations in A = 2 for the notions of filters, ultrafilters, compact spaces ; we also get new equations for the notion of a point.

It is a fact that we cannot get structural equations in 2 for the notion of a reflexive relation and for that of a topological space . However, these notions admit s.e. for $\vec{U}_3$.

So, it is natural to try to measure "complexity" of theories according to the invariant

$$\delta(T) = \inf \{ n \ / \ T \text{ admits a structural equation for } \vec{U}_n \}.$$

III. RELATED FUNCTORS.- Let U be an i.m. over a category C. If RX is the set $\mathrm{Hom}_C(X, FX)$ the function $R : C_0 \longrightarrow Sets$ can be extended to a contravariant functor R^- from C to *Sets* and also to a covariant functor R^+ from C to *Sets*.
In the same way the function $EX = \{ r \in RX \ / \ E(r) \}$ where the equation $E(r) \equiv$ " $r = Fr.i_{FX}.r$ " is a structural equation in 2 of the notion of an equivalence relation, gives rise to a contravariant functor E^- from C to *Sets*.
If E is a structural equation of the notion of a point, and if U satisfies some equations related to E, we can define a functor $V_E : C \longrightarrow Sets$, which assigns to each $X \in C_0$ the set $V_E X$ of "points" of X ("points" being solutions of E in U).

More generally, we can obtain functors V_E^+ or V_E^- for E element of a large class of equations (the functors R^+, R^- and E^- are of this form V_E for some E).

Now, if we assume some supplementary properties to be satisfied by some V_E we obtain more precise theories than the theory of c.i.m., and these theories are of course a better approximation of the theory of topoi.

In order to find which supplementary properties are interesting, we can look at the V_E as candidates for "concrete functors", or for "deductive categories", or perhaps for functors defining "dogmas".

The method of structural equations may, of course, be used in different contexts than the one of involutive monads. We could start, for examples, with (in the *Sets*' case) the functor $P(E^2)$ instead of the functor $P(E)$, and develop a parallel theory. We call "*typical system*" such a context (cf. ([5])).

I would like to conclude this talk by the following question :

Let $E(f_1, \ldots, f_n)$ be an equation satisfied in U_2. Of course E is not necessarily satisfied in each involutive monad. But, if T is an elementary topos and $U(T)$ the canonical involutive monad $\Omega^{(-)}$ over T, is it true that E is satisfied in $U(T)$?

([0]) Esq. Math. Paris VII, vol. 1 (June 70).
([1]) C.R.A.S. Paris, t 275 (July 72) p. 259.
([2]) C.R.A.S. Paris, t 277 (Nov. 73) p. 935.
([3]) C.R.A.S. vol. to appear (2 notes presented on the 1^{st} and 8^{th} of July 74).
([4]) Monades involutives complémentées, to appear in "Cahiers top. et géo. diff.".
([5]) Systèmes typiques (in preparation).