Involutive Monads and Topologies (*)

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At the Sussex' meeting in 74 I gave a talk about the idea of an involutive monad (i·m·) on a category, and I gave a "preparation theorem for duality" [1]. Last year here [2] I explained how one can obtain equational translations of set theoretical notions in the context of i·m . This context is more general than the context of topos and - in my opinion - is more simple. See [1] or [2] for example

Starting with a topos $\underline{E}$ one can express in it essentially three important classes of examples:

- Iff $(\underline{O},\alpha,\leqslant) = O$ is an abelian sup-monoid in $\underline{E}$ [1], then we have an i·m of the form $(O^{(-)},i,\psi)$.
- If j is a topology in $\underline{E}$, we get an i·m $(\Omega_j^{(-)},i,\psi)$ in $\underline{E}$.
- If $A \in \underline{E}_o$, there is a canonical i·m·$((\Omega^A)^{(-)},i,\psi)$ in $\underline{E}$.

We can remark that in some sense the constructions of $\Omega_j^{(-)}$ and $(\Omega^A)^{(-)}$ are the converse of each other [4].

The purpose of this talk is to express a natural context in which we can define the analogue of the transformation $(\Omega,j) \to \Omega_j$ in a topos. For convenience we make the following definition.

Definition:

If C is a monoidal category a cantorian object in C is a triple (A,i,ψ) such that:

- A is an object of C.
- $A^{(-)}$ exists and, for all $x \in C_o$ the canonical map $X \xrightarrow{t_X} A^{(A^X)}$ factorizes as $X \xrightarrow{i_X} A^X \xrightarrow{\psi_X} A^{A^X}$ in such a way that $(A^{(-)},i,\psi)$ is an i·m· on C.

So now the solution may be expressed as follows:

1) If A is a cantorian object in a category C we can define a topology j on A (which corresponds to a topology on Ω in the case of topos).

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2) If j is a topology on A, it is possible (if j splits) to construct a new cantorian object Aj (which corresponds to $(\Omega_j^{(-)}, i, \psi)$ in the case of topos).

In fact, starting with an arbitrary involutive monad U = (F,i,ψ) and a "complementation" on U (in the sense of [ ]), we can say what is meant by a topology j on (U,$\vee$), and by U_j. Precisely U_j turns out to be a quotient of U in the category IM(C) of i.m. on C.

Essentially, this is possible because if U= (F,i,ψ), if X,Y $\in C_o$,

and if $X \overset{R}{\underset{S}{\rightrightarrows}} FY$ are maps, we can <u>define</u>

$R \underset{Y}{\leq} S \Longleftrightarrow (\exists h: X \to F^2Y)(\Lambda_Y h=R$ et $V_Y h=S)$ with

$V_Y = F(t_Y)\psi_{FY}$ and $\Lambda_Y = \vee_Y V_Y \underline{P}(\vee_Y)$ with

$P(\vee_Y) = Fi_{FY} F^2\vee_Y \psi_{YY}$.

N.B. In general this is <u>not</u> an order relation; but in the case of the canonical i.m.$(\Omega^{(-)},i,\psi)$ on topos, it is the usual order on Ω^Y.

Then we define a <u>topology on</u> (U,$\vee$) as a natural transformation j: F$\to$F such that, for each X $\in C_o$,

1) $j_X^2 = j_X$.

2) $Id_{FX} \underset{X}{\leq} j_X$.

3) j_X preserves the relation $\leq_X$.

In the context of (U,$\vee$), 2) and 3) make sense.

[1] Monades involutives complémentées, Cahiers top géo.diff., volume XVI,1 (1er trimester 1975)

[2] Talk at Oberwolfach in 74

[3] Traduction équationnelle de notions ensemblistes, C.R.A.S tome 279 (1974).

[4] Relations continues (to appear).