FIBRATIONS, DIAGRAMS AND DECOMPOSITIONS (*)

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Everyone knows the "equation": $\frac{CAT}{Posets} = \frac{Set^{(-)^{op}}, \text{ Yoneda}, \ldots}{2^{(-)^{op}}, \ldots}$; here we get a solution for $\frac{CAT}{Set} = \frac{?}{2^{(-)}, -, \ldots}$.

For a category X, let $\mathcal{D}X$ be the 2-comma category associated to the situation $1 \xrightarrow{\ulcorner X \urcorner} CAT \longleftarrow Cat$, let $\mathcal{D}X$ be the category of 1-morphisms of $\mathcal{D}X$, and let $d_X : \mathcal{D}X \longrightarrow Cat$ be defined by $(A \xrightarrow{p} X) \longmapsto A$. If $\varphi : \hat{Cat} \longrightarrow Cat$ is the universal fibration, the pullback $d_X^*(\varphi)$ is denoted by $\mathcal{D}^*X$. The funda--mental fact is that

(*) $1 \xrightarrow{\ulcorner 1 \urcorner} Cat$ is 2-dense.

From this it follows that if $K : CAT/Cat \longrightarrow CAT$ is the Grothendieck construc--tion we get

(**) $K \dashv d$ (a 2-adjunction),

and the comonad on CAT associated to this adjunction is $\mathcal{D}^*$.

1°) It can be proved that a coalgebra of $\mathcal{D}^*$ is just a fibration $X \xrightarrow{f} B$ starting from X (this yields a presentation $X = \int_B f$). So, fibrations <u>seen as structures on their total spaces</u> play in CAT the role of equivalences in Set.

2°) In the same way that $(2^X, \subseteq)$ is an order-completion of X, it is true that $\mathcal{D}X$ is a <u>strong-lax-completion</u> of X (a s.l. colimit being a lax- colimit such that there is also the property of unique extension of modifications). As a consequence $\mathcal{D}$ is a monad up to isomorphisms (and $\pi_0^*\mathcal{D}$ is the colimit monad of Kock) with a multiplication $K_X : \mathcal{D}^2X \longrightarrow \mathcal{D}X$, for all X.

(*) Abstract of a talk at the "Category Theory Meeting" at the Isle of Thorns, Sussex, England, on July 25/31 1976.

3°) (Recall : see Guitart, Amiens Meeting 73) The functor $\mathcal{D}$ classifies machines i.e. spans $X \xleftarrow{f} A \xrightarrow{g} Y$ where f is a cofibration.

So, we get a dictionary from Set notions to CAT notions :

Set	CAT
$A \subseteq X$, $P X = 2^X$, $\sup_X : P^2X \to PX$	$A \xrightarrow{p} X$, $\mathcal{D}X$, $K_X : \mathcal{D}^2X \to X$
relation $X \to PY$ or $X \leftarrow A \to Y$	family of diagrams $X \to \mathcal{D}Y$ or machines $X \xleftarrow[fib]{} A \to Y$.
power set monad	diagram monad(up to isomorphisms)
poset $\mathbb{P}X = (PX, \subseteq)$	2-category $\mathbb{D}X$
$\mathbb{P}X$ is a cocompletion of X	$\mathbb{D}X$ is a s.l. cocompletion of X
comonad $P^{\cdot}$ of pointed subsets	comonad $\mathcal{D}^{\cdot}$ of pointed diagrams
equivalence on X (= coalgebra of $P^{\cdot}$ at X)	fibration under X : $X \to B$ (=coalgebra of $\mathcal{D}^{\cdot}$ at X)

Remark 1.

All these results may be localised by pulling back along an arbitrary functor $\Gamma : \underline{M} \to \mathrm{Cat}$ such that $M \in |\underline{M}|, \Gamma(M) \neq \emptyset$.

For instance, the fact that Set is a topos is just a convenient localisation of (**), just as the theory of decompositions of groups is just a convenient localisation of (**) (by the functor Γ : Groups $\rightarrowtail$ Cat).

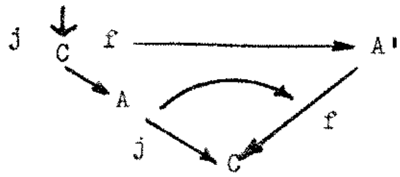
Remark 2.

All these results in fact follow just from the lemma (**), which itself follows from (*), which is nothing more than the 2-Yoneda lemma. It is possible to obtain a version of these results at the n-level, starting with the n-Yoneda lemma :

(n-YL): In n-CAT the morphisme $1 \xrightarrow{\ulcorner 1 \urcorner} (n-1)\text{-Cat}$ is n-dense.

Remark 3.

More generally, the machinery works for an arbitrary category $\underline{C}$, a full subcategory $\underline{A}$, an oject $A \in |\underline{A}|$, a morphism $A \xrightarrow{j} C \in \underline{C}$ and a functor $\underline{A}^{op} \xrightarrow{\mathfrak{C}} \mathcal{C}at$ such that $|-| \circ \mathfrak{C} = \underline{C}(-, C)$. Then the morphism j is said to be $(A, \mathbb{C})$-dense if for $A' \in |\underline{A}|$ and $f : A' \longrightarrow C$ we get a Kan extension (rel. to the structure $\mathbb{C}$)



For instance if $\mathbb{D}$ is a 2-category and D the category of 1-morphisms in $\mathbb{D}$, the axioms of a <u>cosmos</u> (in the sense of Street) may be described as the existence of a certain dense morphism $1 \xrightarrow{t} D$ in Fib(D).

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