TOPOGENESIS AND CONTINUOUS RELATIONS (*)

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1. Because of the lack of time the talk itself will be concerned only with the two first parts of our work entitle

" TOPOLOGIE DANS LES UNIVERS ALGÉBRIQUES.
I–Contextes relationnels et univers algébriques.
II–Topogenèses et types de continuités pour les relations dans un topos.
III– Univers fibrés et illustrations variées."

So, the illustrations about varieties, fuzzy homotopy, compactness, etc..., are postpone and will appear in the final version of the work. More exactly here we would like solely to explain from our work the following

THEOREM.– Let C be a category and $U = (F, i, \psi, \tau)$ a relational context on C ;
– Let $M = (M, \leqslant)$ be an internal complete inf-lattice object in C ;
– Let $\mathcal{T}$ be a topogenesis on C of type (F, M).
Then, the category of U-relations in C τ-continuous between $\mathcal{T}$-spaces is isomorphic to the Kleisli category of a monad $\mathbb{P}$ on $\mathcal{T}$, which is a lifting along the forgetful functor $\theta : \mathcal{T} \longrightarrow C$ of the monad $\mathbb{P} = (P, i, S)$ on C associated to U.

(For the notations see below)

2. Before to elucidate the ingredients of the theorem, it can be observed that the theorem can be employed in the case where C is a topos, where

$$M = (\Omega, \leqslant),\ F = \Omega^{(-)},\ i(a) = \{a\},\ \psi A = \{A' / \exists x (x \in A \wedge x \in A')\},$$

and where $\mathcal{T}$ is any definition of an internal topology in C (by the way of open sets, or closed sets, or by the way of neighbourhoods, etc...) as described in the thesis of Stout. In this case if we take $\tau = \psi$ (resp. if we take $\tau = \pi$ with $\pi A = \{A' / \forall x (x \in A' \rightarrow x \in A)\}$) we get the theory of lower (resp. upper) continuity for relations between $\mathcal{T}$-spaces. So if C = Sets, we get the two classical notions, as shown in ([1]). But if C $\neq$ Sets, because of the fact that the map $\tau \longrightarrow \exp_{\tau}$ is injective in the case where $s = 1$ i.e. $\mathcal{T} = \mathcal{S}^{-}$ (see the §5), there are more notions of continuities in C than there are notions of transpositions on $U = (\Omega^{(-)}, \{-\}, \psi)$ (see §3).

On an other hand, if we take C = Sets and $L = (L, \otimes, \sup)$ a complete abelian monoid $\neq 2$, $ixx' = e$ (resp. o) if $x = x'$ (resp. $x \neq x'$), $\psi_X pp' = \sup_X \{px \otimes p'x\}$, $\pi_X pp' = \inf_X \{p'x \xrightarrow{\otimes} px\}$, $F = L^{(-)}$ and $M = 2$, then we get the various theories of L-to[illegible] L-[illegible] (for def[illegible] the final version [illegible] or ([5])). For instance, in this case, if $* : L \times L \longrightarrow L$ is a binary law, $\tau_X = \sup_X p * p'$

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is a transposition (in the sense of §3) on $(L^{(-)}, i, \psi)$ if and only if

- $\forall\, a \in L$, $a * o = o$ and $a * e = a$.
- $\forall\, a, b, c \in L$, $(a * b) * c = a * (b * c)$
- $\forall\, a \in L$, the maps $a * (-)$ and $(-) * a$ are sup-compatibles.

Such a law will be called a <u>pseudo-tensor</u> on L, and pseudo-homs are defined dually. The last remark is available in fact if we start with a topos, and allows us to see that there are more notions of continuity in a topos (for a given internal notion of topology) than there are sup-endomorphisms of the Heyting algebra Ω .

Therefore the purpose of our theorem is <u>to classify the "convenient" theories of topologies and of continuities for relations between spaces</u> in an arbitrary topos or even in a more general context, in such a way to cover also the various notions of fuzzy continuity, the different calculus of "general paratopology" (following the expression of Coppey), and of course the ordinary theory of Moore closure(see also the works of Gogen, Sols, Zadeh). In order to provide such a classification it is necessary to classify successively the notions of "relations", of "transposed relation" and of "topology".

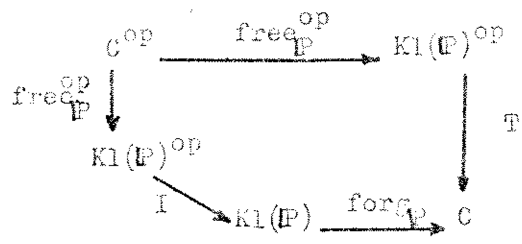
3. For the first step we embrace the notion of an <u>involutive monad</u> of ([2]); if we add to an involutive monad (Let us recall that such an involutive monad is seen as playing the rôle of a formal "monad of subsets") (F, i, ψ) on C a <u>transposition</u> τ we get a <u>relational context</u>, the datas of which are :

- a functor $F : C^{op} \longrightarrow C$ (like $\Omega^{(-)}$ in §2),
- for each X in C, three maps $i_X : X \longrightarrow FX$, $\psi_X : FX \longrightarrow F^2X$ and $\tau_X : FX \longrightarrow F^2X$,

(for examples see §2, and ([2])), these datas beeing submitted to seven equational conditions, in such a way that a relational context admits the alternative description :

- a monad $\mathbb{P} = (P, i, S)$ on C,
- an involution $I : Kl(\mathbb{P})^{op} \longrightarrow Kl(\mathbb{P})$ on the Kleisli category of $\mathbb{P}$,
- a transposition on $(\mathbb{P}, I)$, i.e. a functor $T : Kl(\mathbb{P})^{op} \longrightarrow C$ such that

the diagram



commuts.

If we begin with T, we get $\tau_X = T(S_X)$, and if τ is given, the transposed of a $\mathbb{P}$-relation (i.e. of a morphism $R : X \longrightarrow PY = FY$ in C) is given by the formula

$$T(R) = R^{\tau} = F(R).\tau_Y.$$

One can observed that the data of a transposition is essentially equivalent to the data of a formal adjonction calculus (for that, if $R : FX \to FY$ and $S : FY \to FX$ are maps in C, we define "$S \dashv R$" has being "$\tau_Y . R = F(S) . \tau_X$").
Finally, although it is not necessary for our theorem, an algebraic universe is a category with cartesian products, and equipped with a relational context related to the cartesian structure by the data of a family of maps $c_X : X^2 \to FX$ (playing the rôle of $(x,y) \mapsto \{x, y\}$) submitted to several equational conditions. So, formally, the ingredients of an algebraic universe take place in a 9-tuple $U = (C, \times, pr_1, pr_2, F, i, \psi, \tau, c)$.
PROPOSITION.- Any sentence of the usual first order language of mathematics has a canonical equational translation in an arbitrary algebraic universe.

4. For the second step we have to defined a topogenesis of type (F, M).
For that let $\mathcal{S}^-$ be the pullback

$$\begin{array}{ccc} \mathcal{S}^- & \longrightarrow & (C//M)^{op} \\ {\scriptstyle |-|}\downarrow & & \downarrow \\ C & \xrightarrow{F^{op}} & C^{op} \end{array}$$

We let $\Sigma(M)$ be the category (over C) of continuous maps between Moore closure operators on lattices of the form $Hom_C(X, M)$.
PROPOSITION.- The following datas are equivalent :

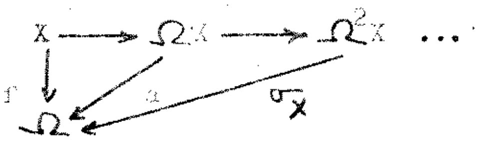
(i) A factorisation s of F through $\Sigma(M) \longrightarrow C$

$$\begin{array}{ccc} & \xrightarrow{s} & \Sigma(M) \\ C^{op} & & \downarrow \\ & \xrightarrow{F^{op}} & C \end{array}$$

(ii) A full subcategory $\mathcal{T}$ of $\mathcal{S}^-$ coreflexive over $|-|$.

Such a data is called a topogenesis of type (F, M), and the functor $\mathcal{T} \rightarrowtail \mathcal{S}^- \xrightarrow{|-|} C$ is denoted by $\theta : \mathcal{T} \longrightarrow C$. An object of $\mathcal{T}$ is a couple (X, a) where $a : FX \to M$ is such that $s_X(a) = a$, and is called a $\mathcal{T}$-space.
The notion of topogenesis is not very different from the wyler'notion of a Top-category. Our definition here insists on the fact that a topogenesis is a kind of presentation "by generators and relations" of a notion of topological space (see examples in §2). It is also interesting to observe that the Grothendieck' topologies are "representable" examples of topogenesis on a topos.
If s (or $\mathcal{T}$) is a topogenesis on a topos, of type $(\Omega^{(-)}, \Omega)$, s is defined by by the caracteristic maps $\sigma_X : \Omega^2 X \to \Omega$ of s-closed families, any object of $\mathcal{T}$ is [illegible] : $\Omega X \longrightarrow \Omega$, [illegible] any closed subobject of X is a special [illegible] $f : \ldots \to \Omega$

and this yields to the beginning of a simplicial object in C :



Now if a : FX ⟶ M and b : FY ⟶ M are objects in $\mathcal{T}$ (i.e. $\mathcal{T}$-spaces) a U-relation R : X ⟶ FY is called τ-<u>continuous</u> between the $\mathcal{T}$-spaces a and b if we have

$$b \leqslant a.R^{\tau} \quad (= a.F(R).\tau_Y).$$

5. For the last step we have to mixed the ingredients:
Firstly, for each map f : X ⟶ Y we define $\exists_M(f) : Hom_C(X, M) \longrightarrow Hom_C(Y, M)$ by

$$\exists_M(f)(p)(y) = \sup_X \{ p(x) \,/\, f(x) = y \}.$$

Then we define the lifting $\overline{P}$ of P along θ by

$$\overline{P}(X, a) = (PX,\ s_X(\exists_M(\tau_X)(a))),$$

for each $\mathcal{T}$-space a : FX = PX ⟶ M.

References

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