1. Because of the lack of time the talk itself will be concerned only with the two first parts of our work entitle

"TOPOLOGIE DANS LES UNIVERSES ALGÉBRIQUES.
I-Contextes relationnels et univers algébriques.
II-Topologie et types de continuïté pour les relations dans un topos.
III-Univers fibrés et illustrations variées."

So, the illustrations about varieties, fuzzy homotopy, compactness, etc..., are postpone and will appear in the final version of the work. More exactly here we would like solely to explain from our work the following

THEOREM - Let $C$ be a category and $U = (F, i, \rho, \tau)$ a relational context on $C$;
- Let $u = (u, \leq)$ be an internal complete inf-lattice object in $C$;
- Let $U$ be a topos schema on $C$ of type $(F, k)$.

Then, the category of $U$-relations in $C$ $\tau$-continuous between $\mathcal{C}$-spaces is
determined to the Alchourron category of a monad $\mathcal{M}$ on $\mathcal{C}$, which is a lifting along
the forgetful functor $\theta : \mathcal{C} \to C$ of the monad $\mathcal{M} = (F, i, \rho)$ on $C$ associated to $u$.

(For the notations see below)

2. Before to elucidate the ingredients of the theorem, it can be observed that the theorem can be employed in the case where $C$ is a topos, where

$\mathbb{K} = (\mathbb{L}, \leq)$, $F = \mathbb{L}^{(\sim)}$, $i(a) = \{a\}$, $\psi : \mathbb{A} \to x(x \in \mathbb{A} \land x \in \cdot^a)$
and where $\mathcal{C}$ is any definition of an internal topology in $C$ (by the way of
open sets, or closed sets, or by the way of neighbourhoods, etc...) as described in the thesis of Stout. In this case if we take $\tau = \psi$ (resp. if we take $\tau = \pi$
with $\pi : \mathcal{A} \to \{x(x \in \mathbb{A} \to x \in \cdot^a)\}$) we get the theory of lower (resp. upper) continuity for relations between $\mathcal{C}$-spaces. So if $C = \text{Sets}$, we get the two
classical notions, as known in (1). But if $C \neq \text{Sets}$, because of the fact that the map $\tau \to \exp_\tau$ is injective in the case where $\rho = 1$, i.e. $\mathcal{C} = \mathcal{C}^\tau$ (see the §3), there are more notions of continuities in $\mathcal{C}$ than there are notions of
transpositions on $U = (\mathbb{L}^{(\sim)}$, $[-\cdot], \psi)$(see §3).

On the other hand, if we take $C = \text{Sets}$ and $L = (L, \varnothing, \text{sup})$ a complete lattice
monoid / 2, $i(x \cdot^a = e$ (resp. $o$) if $x = x^a$ (resp. $x \neq x^a$), $\psi : \mathcal{A} \to \text{Sup}_{\mathbb{A}}\{p \cdot^x \mathcal{A} \}^\mathcal{A}$,
$x \cdot^a = \text{Inf}\{x^a : \mathbb{A} \to \mathcal{A}\}$, $\rho = L^{(-)}$ and $k = 2$, then we get the various theories of
$L$-approximation and closure notions. (For details - the §4, version 1, 2 or (2)). For instance, in this case, if $\mathcal{M} : \mathcal{L} \to \mathcal{L}$ is a binary law, $\mathcal{M} = \text{Sup}_{\mathcal{A}}\mathcal{A}$.
is a transposition (in the sense of \(\mathcal{S}\)) on \((L^\mathcal{S}, i, \mathcal{P})\) if and only if
\[
\forall x \in L, \ a \circ x = a \quad \text{and} \quad b \circ x = b.
\]
\[
\forall x, y, z \in L, \ (x \circ y) \circ z = x \circ (y \circ z)
\]
\[
\forall x \in L, \ \text{the maps} \ a(\mathcal{S}) \text{ and} \ a(\mathcal{S}) \circ x \text{are sup-composites}.
\]
Such a law will be called a pseudo-tensor on \(L\), and pseudo-homs are defined dualy.

The last remark is available in fact if we start with a topos, and allows us to see that there are more notions of continuity in a topos (for a given internal notion of topology) than there are sup-endo morphisms of the Heyting algebra \(\mathcal{O}\).

Therefore the purpose of our theorem is to classify the "convenient" notions of topologies and of continuities for relations between spaces in an arbitrary topos or even in a more general context, in such a way to cover also the various notions of fuzzy continuity, the different calculus of "general paratopology" (following the expression of Goyon), and of course the ordinary theory of Moore closure (see also the works of Goguen, Soli, Zidek). In order to provide such a classification it is necessary to classify successively the notions of "relations", of "transposed relation" and of "topology".

For the first step we embrace the notion of an involutive monad of \((\mathcal{S})\); if we add to an involutive monad (let us recall that such an involutive monad is seen as playing the role of a formal "monad of subsets") \((\mathcal{P}, i, \mathcal{P})\) on \(C\) a transposition \(\tau\) we get a relational context, the data of which are:

- A functor \(\tau: C \to C\) (like \(\mathcal{L}^\mathcal{S}\) in \(\mathcal{S}\)),
- For each \(X\) in \(C\), three maps \(1_X: X \to FX, \mathcal{P}_X: FX \to F^2 X\) and \(\tau_X: FX \to F^2 X\).

(for example see \(\mathcal{S}\) and \((\mathcal{S})\)), these data being submitted to seven equational conditions, in such a way that a relational context admits the alternative description:

- A monad \(T = (\mathcal{P}, i, S)\) on \(C\),
- An involution \(\mathcal{I}: K_1(\mathcal{P})^{\mathcal{O}} \to K_1(\mathcal{P})\) on the Kleisli category of \(\mathcal{P}\),
- A transposition on \((\mathcal{P}, i, I)\), i.e., a functor \(T: K_1(\mathcal{P})^{\mathcal{O}} \to C\) such that

the diagram

\[
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\text{freq}} & K_1(\mathcal{P})^{\mathcal{O}} \\
\downarrow{\text{freq}} & & \downarrow{T} \\
K_1(\mathcal{P})^{\mathcal{O}} & \xrightarrow{\mathcal{I}} & C
\end{array}
\]

commutes.

If we begin with \(T\), we get \(\tau_X = T(S_X)\), and if \(\tau\) is given, the transposed of a \(\mathcal{S}\)-relation (i.e., of a relation \(S_X \subseteq FX \times FY\) in \(C\)) is given by the formula

\[
T(\tau) = \tau = \tau_X \cdot \tau_Y.
\]
One can observe that the data of a transposition is essentially equivalent to the data of a formal adjunction calculus (for that, if \( H : F \to G \) and 3 : FX \to FY are maps to C, we define "3 \to R" has being "\( \mathcal{T}_X \Ra R \to \mathcal{T}_X \Ra S \)"").

Finally, although it is not necessary for our theorem, an algebraic universe is a category with cartesian products, and equipped with a relational context related to the cartesian structure by the data of a family of maps \( c \times X \to FY \) (playing the role of \( (x, y) \mapsto \{ (x, y) \} \)) submitted to several equational conditions. So, formally, the ingredients of an algebraic universe take place in a 9-tuple

\[
\mathcal{H} = (C, X, x, \psi, \tau, \xi, \nu, \rho, p, r, \mathcal{T}, \delta, \Omega).
\]

**Proposition.** Any sentence of the usual first order language of mathematics has a canonical equational translation in an arbitrary algebraic universe.

4. For the second step we have to defined a topogenesis of type \((P, H)\).

For that let \( \mathcal{J}^- \) be the pullback

\[
\begin{array}{ccc}
\mathcal{J}^- & \longrightarrow & (C/H)^{\text{op}} \\
\downarrow \text{1-1} & & \downarrow \text{1-1} \\
C & \longrightarrow & C^{\text{op}}
\end{array}
\]

We let \( \mathcal{E}(X) \) be the category (over \( C \)) of continuous maps between Moore closure operators on lattices of the form \( \text{Hom}_C(X, H) \).

**Proposition.** The following data are equivalent:

1. A factorization \( s \) of \( F \) through \( \Sigma(X) \to C \)

\[
\begin{array}{ccc}
\Sigma(X) & \longrightarrow & C \\
\downarrow \text{1-1} & & \downarrow \text{1-1} \\
C & \longrightarrow & C^{\text{op}}
\end{array}
\]

2. Full subcategory \( \mathcal{C} \) of \( \mathcal{J}^- \) coreflective over \( 1-1 \).

Such a that is called a **topogenesis of type \((P, H)\)**, and the functor \( \mathcal{C} \to \mathcal{J}^- \)

is denoted by \( \mathcal{B} : \mathcal{C} \to \mathcal{J}^- \). An object of \( \mathcal{C} \) is a couple \((X, a)\) where \( a : F \to X \)

such that \( \mathcal{C}(a) = a \), and is called a **\( \mathcal{C} \)-value**.

The notion of topogenesis is not very different from the wyler\'s notion of a top-category. Our definition here insists on the fact that a topogenesis is a kind of presentation "by generators and relations" of a notion of topological space (see examples in \( \mathcal{J} \)). It is also interesting to observe that the Grothendieck toposes are "representable" examples of topogenesis in a topos.

If \( \mathcal{C} \) is a topogenesis on a topos \( \Omega \), then \( \mathcal{C}(\mathcal{J}^- \Omega, \Omega) \), is defined by

by the characteristic maps \( \mathcal{X} \to \Omega \) of \( \mathcal{C} \)-closed families, any object of \( \mathcal{C} \) is

\[
\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \Omega \\
\downarrow \text{1-1} & & \downarrow \text{1-1} \\
\Omega & \longrightarrow & \Omega
\end{array}
\]

\( \mathcal{C} \)-closed subobject of \( X \) in \( \mathcal{C} \) a special form \( \Omega \).
and this yields to the beginning of a simplicial object in $\mathcal{C}$:

$$
\begin{array}{c}
\xymatrix{
X & \Omega X & \Omega^2 X & \cdots \\
\sigma & \ar[ur] & & \\
\Omega_{2} & & & \\
}\end{array}
$$

Now if $a : PX \rightarrow \mathbb{N}$ and $b : PY \rightarrow \mathbb{N}$ are objects in $\mathcal{C}$ (i.e. $\mathcal{C}$-spaces) a $U$-relation $R : X \rightarrow Y$ is called $\mathcal{C}$-continuous between the $\mathcal{C}$-spaces $a$ and $b$ if we have

$$
b \leq a \ast R \quad (= a \ast P(R) \ast R).$$

For the last step we have to mixed the ingredients:

1. Firstly, for each map $f : X \rightarrow Y$ we define $\exists_X(f) : \text{Hom}_{\mathcal{C}}(X, \mathbb{N}) \rightarrow \text{Hom}_{\mathcal{C}}(Y, \mathbb{N})$ by

$$
\exists_X(f)(p)(y) = \sup \{ p(x) \mid f(x) = y \}.
$$

Then we define the lifting $F$ of $P$ along $\theta$ by

$$
F(X, a) = (PX, \varepsilon_X(\exists_X(\theta))(a)),
$$

for each $\mathcal{C}$-space $a : PX \rightarrow \mathbb{N}$.

### References


Université PARIS VII

Département de Math

Tours 45-20, 2ème étage,

2, place Jussieu,

75005, PARIS.