A L G E B R A I C   U N I V E R S E S

R e n é   G U I T A R T

Lecture at the Summer School
on Universal Algebra and Ordered Sets
Jindřichův Hradec, september 1979

I. MOTIVATIONS AND RESTRICTIVE REMARKS.

1. There is no privilegiate "universe" in which to do mathematics. And particularly to work in models of set theory is not enough. This fact is more or less clear in the mind of mathematicians since about 25 years. For different parts of mathematics we need different kinds of foundations. Examples :

Topos theory of Grothendieck	$\longrightarrow$ for Algebraic Geometry
Non-standard model theory of Robinson	$\longrightarrow$ for Infinitesimal Analysis
Boolean set theory of Scott-Solovay	$\longrightarrow$ for Set Theory
Fuzzy set theory of Zadeh	$\longrightarrow$ for Automata Theory

For the moment, let us observe just that in all these cases "there is in question " a category plus an additional structure. This lead us to think that we need to do mathematics in very general categories.

2. In order to do algebraic theories in a category C the situation now is clear enough (after Grothendieck, Ehresmann, Chevalley, Lawvere, Benabou,

Linton, etc.): essentially we need on C a calculus of $\varprojlim$. Another way to express that is to say that algebraic theories are described by <u>projective sketches</u>. In the case of an algebraic theory with only everywhere defined operations, we can work also from the point of view of monoïdal categories.

3. But now the question is : what is the material on a category C which is usefull in order to express general topology, and, more systematically, all first order theories ? The answer is that, after the calculus of lim (or eventually the calculus of $\otimes$), we have to give us <u>something in order to interpretate the "logic"</u>. For that, we have two main possibilities :

(a) - the calculus of $\varinjlim$ (i.e. the theory of (mixed) sketches).

(b) - the calculus of relations.

In this lecture I'll not speak of (a) and of the interactions between (a) and (b). I think that it will be interesting to study these points, and probably I'll do it in a next futur. Now I would like to concentrate my attention on (b) alone.

3. In the case of the category of sets, the interpretation of the logic is possible because of the existence in Set of the object

$$2 = \{o, 1\} = \{\text{true, false}\}$$

So, for each set X we have

$$P\,X = 2^X = \{A \subset X\}$$

the set of all subsets of X, and then the "logical calculus" is the calculus of maps like $Y \xrightarrow{r} P\,Y$, i.e. $\underline{r} \subset Y \times X$,

i.e. <u>the calculus of relations with the help of P</u>.

4. In the case of the category Top of topological spaces, for each space X we have

$$P\,X = \{A \subset X\} + \text{finite top. of Michael-Choquet.}$$

5. In the case of the category of manifolds Man, the situation is difficult : for a manifold V, it is not clear if Sub V - the set of submanifolds of V - has a structure of manifold. There are works on this subject by Riemann, Douady, Cerf.

In the sequel of this lecture we'll retain the idea that a universe (that is to say some place in which to do mathematics) consists of :

$$C \text{ (category)} + \underleftarrow{\lim} + \mathbb{P} + ? + ? \ldots$$

The aim of the lecture is <u>to precise what I mean by that</u> (in such a way to cover examples) <u>and to explain how to do maths inside it</u>.

<u>Rm 1</u>. In fact my point of view will appear as a kind of categorization of the ideas of Bourbaki-Ehresmann about <u>type-functors</u> and structures.

<u>Rm 2</u>. In the theory of relations a crucial fact is the role play by <u>exact squares</u> as I have shown in "Relations et Carrés exacts" (to appear) ; because of the lack of time I have decided to <u>not</u> speak here of my general theory of exactness in a representable 2-category, and so for this question I send back to this forthcoming paper.

<u>Rm 3</u>. I shall not speak here of the elementary toposes of Lawvere-Tierney except that they are examples of algebraic universes. On the difference of quality of there leading philosophies, I would like just to write down the informal proportion

$$\frac{\text{Algebraic Universes}}{\text{Topos}} = \frac{\text{Algebras}}{\text{Fields}}$$

<u>Rm 4</u>. In despit of their generality, the algebraic universes doesn't covered all the examples we have in mind : so, regular categories or pre-toposes are <u>not</u> examples of algebraic universes, but they are examples of <u>pro-algebraic universes</u> (i.e. the monad $\mathbb{P}$ exist only as a pro-monad or a monad on $\mathrm{Ens}^{C^{op}}$). In this lecture I shall <u>not</u> go on pro-ification of the ideas, and I'll always suppose that each object of which I am speaking exist really.

<u>Rm 5</u>. As I said in Rm 4, I'll work only in <u>the representable case</u> ; and more, I'll stay in <u>the discrete case</u>. By "the discrete case" I mean that the base category C play the role of Ens (the category of sets) in such a way that its objects are discrete (as the sets are discrete categories) . The non-

discrete structure of C would be observable by the existence on C of a non-trivial involution (as for example the involution $(-)^{op} : Cat \longrightarrow Cat$ in the case $C = Cat$). In these non-discrete cases, the idea of an involutive monad would be replaced by the idea of a G-<u>involutive monad</u> (where G is an automorphism group of C) i.e. a monad $\mathbb{P}$ on C plus, for all $g \in G$, $X \in C_o$, $Y \in C_o$, the datas of bijections

$$I_g^{X,Y} : \mathrm{Hom}_C(X, PY) \longrightarrow \mathrm{Hom}_C(gY, g^{-1}PX)$$

compatible with the composition in G, the composition in C, and the multiplication of the monad.

<u>Rm 6</u>. After all these limitations of the subject, I invite the auditory to just keep in mind the main point : the theory of algebraic universes is an attempt to get an efficient <u>non-classical predicat's calculus</u>.

II. MONADS AND TOPOS.

As the auditory doesn't consist only of specialists in categories, I'll take the time to make some preliminaries, and to say what are monads and what are toposes.

A <u>monad</u> on a category C is a 3-tuple (T, u, m) where $T : C \longrightarrow C$ is a functor, $u : Id_C \longrightarrow T$ and $m : T_\circ T \longrightarrow T$ are natural transformations, and the following equations hold : $m.Tu = m.u_T = Id_T$, $m.Tm = m.m_T$.

For example, each algebraic theory on Ens generate a monad on Ens with

$$T\,X = \text{Free alg. over } X$$

So we have :

1) $T\,X = (X)^* =$ Free monoid over X, $u : x \longmapsto (x)$, $m =$ concatenation.

2) $T\,X = \mathcal{P}X = \{A \subset X\} = 2^X$, $u : x \longmapsto \{x\}$ $m =$ general union.

3) Let $L = (L, \leqslant, \otimes)$ a complete abelian monoid (e.g. $(2, \leqslant, \wedge)$ or $([0, 1], \leqslant, .)$; then we can construct a monad with $T\,X = L^X$, etc.

Let C be a category and $(T, u, m) = \mathbb{T}$ a monad on C. We define the Kleisli category of $\mathbb{T}$ as follow : its objects are the objects of C, and the morphisms in Kleisli($\mathbb{T}$) from X to Y are just the morphisms in C from X to $T\,Y$. If you want, in the case where $T\,X =$ Free alg. over X Kleisli($\mathbb{T}$) is the category of all the homomorphisms between free algebras. In particular in the example 2 Kl $\mathbb{T}$ is the category of relations between sets, and in the example 3, Kl $\mathbb{T}$ is the category of L-fuzzy relations between sets.

In general there is clearly a canonical functor $L^{\mathbb{T}} : C \longrightarrow \mathrm{Kl}\ \mathbb{T}$ bijective on objects ; the morphisms in Kl $\mathbb{T}$ which are not in C "are" the algebraic operations of the algebraic theory over C described by $\mathbb{T}$.

A <u>topos</u> is category C with finite projective limits and with a sub-object classifier that is to say a morphism$(1 \xrightarrow{\text{true}} \Omega)$ C such that for every monomorphism $(Y \overset{m}{\rightarrowtail} X) \in C$ there is exactly one morphism $\mathrm{car}_m : X \longrightarrow \Omega$ such that the diagram

$$\begin{array}{ccc} Y & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & = & \downarrow{\scriptstyle \text{true}} \\ X & \xrightarrow[\mathrm{car}_m]{} & \Omega \end{array}$$

is a pullback.

For example the category of sets, Ens, is a topos, because with $\Omega = 2$, each subset $Y \subset X$ has a caracteristic function $\mathrm{car}_Y : X \longrightarrow 2$, in such a way that sub.obj.$(X) = 2^X$.

More generally, every category of presheaves i.e. of the form $\mathrm{Ens}^{I^{op}}$ and even every category of sheaves on a Grothendieck 'site is a topos (the so called Grothendieck's toposes).

In an arbitrary topos it is possible to construct a "powerset monad" (P, a, S) on the topos with $P\,X = \Omega^X$, with a a singleton map and with S a union map. In the case of the topos Ens we get the powerset monad (example 2).

In 1970, when Lawvere and Tierney had produced their idea of topos, I had begun to study axiomatically such a "powerset monad" ; in 1976 this have been achieved in the theory of algebraic universes. Here, in order to go progressively to the complete list of datas of an a.u., we have to precise the "structure" of the powerset monad. The next step will be the idea of monoïdal monad.

III. MONOÏDAL MONADS, INVOLUTIVE MONOIDAL MONADS.

Roughly speak a monoïdal monad will be a monad associated to a commutative theory. For example the monad $(\ )^*$ of the example 1 is not monoïdal because the transformations $X^* \times Y^* \longrightarrow (X \times Y)^* : ((a_1,\dots,a_n),(b_1,\dots,b_p)) \longrightarrow$

$((a_1,b_1),\ldots,(a_1,b_p),(a_2,b_1),\ldots)$ is not compatible with the concatenation, and this appears because of the lack of commutativity of the theory of monoïds.

A monoïdal monad on a category C consists of a monad (P, a, S) plus a natural transformation $D : P(-) \otimes P(.) \longrightarrow P(-\otimes .)$ (where $\otimes$ is a given monoïdal structure on C) such that, for all $X, Y \in C_o$,

$$S_{X\otimes Y} \cdot PD_{X,Y} \cdot D_{PX,PY} = D_{X,Y} \cdot (S_X \otimes S_Y) \quad \text{and} \quad a_{X\otimes Y} = D_{X,Y} \cdot (a_X \otimes a_Y).$$

When we have a monoïdal monad we get a prolongation $\bar{\otimes} : \mathrm{Kl}\,\mathbb{P}^2 \longrightarrow \mathrm{Kl}\,\mathbb{P}$ of $\otimes : C^2 \longrightarrow C$ given by

$$(X \xrightarrow{r} PY,\ X' \xrightarrow{r'} PY') \longmapsto (X\otimes X' \xrightarrow{r\otimes r'} PY \otimes PY' \xrightarrow{D_{Y,Y'}} P(Y\otimes Y')).$$

An involutive monad on a category C consists of a monad (P, a, S) plus an involution on its Kleisli'category i.e. a contravariant foncteur $I : \mathrm{Kl}\,\mathbb{P} \to \mathrm{Kl}\,\mathbb{P}$ such that $I^2 = \mathrm{Id}_{\mathrm{Kl}\,\mathbb{P}}$ and such that for every $X \in C_o$, $I(X) = X$.

So I produce bijections $I^{X,Y} : \mathrm{Hom}_C(X, PY) = \mathrm{Hom}_C(Y, PX)$ (associating to each relation its "opposite" or "symetric" relation).

Now an involutive monoïdal monad consists of a monoïdal monad ((P, a, S), D) plus an involution I on $\mathrm{Kl}\,\mathbb{P}$ compatible with $\bar{\otimes}$ i.e. such that

$$I.(-\bar{\otimes}\,.\,) = I(-)\,\bar{\otimes}\, I(.).$$

The powerset monad on Ens, the powerset monad on a topos, the monad on Ens given in example 3 (associated to a complete abelian monoïd) are some examples of involutive monoïdal monads. And in fact they are more as w'll see hereunder

IV. THE DATAS OF AN ALGEBRAIC UNIVERSE.

An algebraic universe consists of a category C with finite lim, equipped with an involutive monoïdal monad (w.r.t. the cartesian product $\times$ on C) ((P, a, S), D, I) an with two functors $J, T : (\mathrm{Kl}\,\mathbb{P})^{op} \longrightarrow C$ computing for each relation r its opposite and its adjoint (i.e. in the sets'case, we have $JrB = \{x \,/\, rx \cap B \neq \emptyset\}$ and $TrB = \{x \,/\, rx \subset B\}$) with $U^{\mathbb{P}}.I = J$ (with $U^{\mathbb{P}}$ the canonical forgetful functor from $\mathrm{Kl}\,\mathbb{P}$ to C) and with

$$J.L^{P\,op} = T.L^{P\,op} =: F.$$

We assume also that D can be described from F, S, a and a "link"

$$o_X \quad : \quad X \times X \longrightarrow P\,X \qquad ((x,y) \longmapsto \{x,y\})$$

(in the set case this is possible, and for the continuation of this lecture it is not necessary to be more precise).

Finally for an algebraic universe we assume the "kernel condition" :

for all $X \in C_o$, a_X is a kernel of a_{PX} and Pa_X.

We have a reduction theorem expressing that all the datas of an a.u. (i.e. P, a, S, J, T, D, o) can be expressed from the datas F,o, ψ and τ , with $\psi_X = J\,1_{PX}$ and $\tau_X = T\,1_{PX}$.

The monad (P, a, S) play the role of the powerset monad on Ens, and , with $t_X = \psi_X . a_X$, the 3-tuple $(F.F,\ t,\ Ft_F)$ is a monad on C, playing the role of the double dualization monad $2^{2^{(-)}}$ on Ens.

Now, in order to do mathematics, we can work in Ens, in a topos, in a category L-Ens of L-fuzzy sets, or, more generally, in an a.u.

$$\begin{tikzcd} & \text{Ens} \arrow[dl] \arrow[dr] & \\ \text{Topos} \arrow[dr] & & \text{L fuzzy sets} \arrow[dl] \\ & \text{a.u.} & \end{tikzcd}$$

It is really possible to construct mathematics in an a.u. because of the following result : a first order typified structure functor of Bourbaki-Quine is exactly an object of the finite projective completion of the category generated by the object Id_{Ens}, $\mathcal{P}$, $-\times\cdot$, F, $F_\circ F$, $F_\circ P_\circ F$, etc and all their possible compositions, and by the morphisms which consist of all the natural transformations generated by ψ, τ, o, etc.

Other possibilities exist in order to introduce the structures in an a.u. : so we can describe the compact topologies as the algebras of the ultrafilters monad $\mathcal{U}$, and $\mathcal{U}$ can be obtain as a certain submonad of F^2.

It is also possible to introduce a transfinite aritmetic of "bounds" (a bound is defined just as a subfunctor of P).

In an a.u. the monad $\mathbb{P}$ is monoïdal and because of that the category $C^{\mathbb{P}}$ of algebras of $\mathbb{P}$ is a monoïdal category, as it is proved in my paper "Tenseurs et machines" (to appear), and, consequently, the study of $\mathbb{P}$-fuzzy automatas can be reduced for a big part to the study of deterministic automatas in the monoïdal category $C^{\mathbb{P}}$.

V. ALTERATION OF THE LOGIC IN A GIVEN A.U.

Let $(C \,;\, P, a, S, I, D, c\,) = U$ be an a.u., an let A be an object of C.
I would like to show how it is possible to construct on the same category C another "logic" i.e. another system P', a', S', etc, such that :

i/ $P'(1) = A$

ii/ $(C \,;\, P', a', S', I', D', c') = U'$ is an a.u.

Such a universe U' will be call an <u>alteration</u> of the (logic of the) universe U.
The result is that this construction is possible if the object A is equipped with

- a structure of abelian monoïd $k : A \times A \longrightarrow A$
- a structure of P-algebra $\theta : P(A) \longrightarrow A$

and if k and θ are compatible i.e.

$$\begin{tikzcd} PA \times PA \arrow[r, "D_{A,A}"] \arrow[d, "\theta\times\theta"'] & P(A \times A) \arrow[r, "P\,k"] \arrow[d, phantom, "="] & PA \arrow[d, "\theta"] \\ A \times A \arrow[rr, "k"'] & & A \end{tikzcd}$$

For example the case 3 in § II is an alteration of the classical logic on Ens.

On a topos if j is a Grothendieck topology, the object $A = \Omega_j$ (classifier of j-closed subobject) can be taken to generate an alteration of the canonical logic given by Ω . This alterate logic describe the behavior of j-sheaves. In this alterate logic we identify some logical values. It is also possible to take an arbitrary object B in the topos, and then to take $A = \Omega^B$; the corresponding alterate logic is obtained by copies of logical values.

This result is the real meaning of the Rm 3 § I : analogically starting with a field K one can augment it in a K-algebra of matrices M, and then M can be augmented in M', and so on. So, the class of algebra is "stable" by augmentation as the class of a.u. is stable by alteration.

VI. TOPOGENESIS AND CONTINUOUS RELATIONS.

Roughly speaking we have the following picture :

Universal algebra
$\Vert$
$\left\{\begin{array}{l}\text{Letters}\\ \quad +\\ \text{Grammar } (= \text{equations})\end{array}\right.$
$\Downarrow$
an idea of "language"

Universal topology
or
Topogenesis
$\Vert$
$\left\{\begin{array}{l}\text{Locus}\\ \quad +\\ \text{Incidence relations}\end{array}\right.$
$\Downarrow$
an idea of "space"

$\searrow$ $\swarrow$

To speak on space
(= ?)
To do Mathematics

So the precise definition of a topogenesis have to be such that all the more or less classical notions of space are examples of topogenesis: topological spaces defined by open sets, top.spaces defined by closed sets, top. defined by neigborhoods, uniform spaces, Moore closures, Ehresmann paratopologies, Grothendieck topologies, Choquet pré-topologies, Kowalsky limesraûmes,etc.

An <u>abstract topogenesis</u> on a category C is a data of the form

$$T \underset{i}{\overset{s}{\rightleftarrows}} S \xrightarrow{\ p\ } C$$

where p is a cofibration and s a right adjoint to i with $s.i = Id_T$.

When C is an a.u. we have a calculus of relations on C, and it can be asked for a good lifting of this calculus into the category of spaces T, in such a way to get a calculus of continous relations. To do that, we consider topogenesis on C of type (F, M) where F is the contravariant powerset functor on C, and M is an internal lattice in C ; such a topogenesis is by definition of the form

$$T \underset{i}{\overset{s}{\rightleftarrows}} S^{-}(F,M) \xrightarrow{\ /-/\ } C$$

where $S^-(F, M)$ is a pullback along $F : C \longrightarrow C^{op}$ of the functor $(C//M)^{op} \longrightarrow C^{op}$: so its objects are the (X, a) with $X \in C_o$ and $a : FX \longrightarrow M$, and its morphisms from (X, a) to (Y, b) are the $f : X \longrightarrow Y$ which are "continuous" i.e. such that $b \leqslant a.F(f)$.

A $\mathcal{T}$-continuous relation from an object of T to another is a relation $r : X \longrightarrow PY$ such that $b \leqslant a.F(r).\mathcal{T}_Y$.

By the formula $\bar{F}(X, a) = (FX, s_X(P_M(\mathcal{T}_X)(a)))$ it is possible to describe a monad on T such that the category of $\mathcal{T}$-continuous relations between T-spaces is precisely the Kleisli category of this monad.

As examples when C = Ens we get the lower continuous relations, the upper continuous relations. In the case where C is a topos there are as much types of continuity for relations as there are complete lattice structures on the object Ω (in the internal meaning).

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I would like to thank very much J.ROSICKY and M.SEKANINA for their invitation to lecture at this Summer School. This have been for me a good opportunity to try to give a quick survey of my works. Now, I hope that in a near future I 'll produce a more detailled general survey including also my works on exact squares and on machines.

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