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FROM WHERE DO FIGURATIVE ALGEBRAS COME ?

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1. I have introduced the notion of Figurative Algebra in the paper

(+) Introduction à l'Analyse Algébrique.
II. Algèbres Figuratives et esquisses.

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I have also talked on this paper in the "Séminaire de Catégories de Paris 7" (nov. 1981), in Hagen (nov. 1981), Bremen (dec. 1981), Budapest (march 1982), Genova (April 1982). After several interesting discussions with especially V. Topentcharov, W. Tholen, M. Wischnewsky, H. Andréka, I. Németi, L. Marki, M. Grandis, C. Lair, L. Coppey, R. Brown, A. Burroni, D. Bourn, A. Ehresmann, L. Van den Bril, I have understood progressively that this type of ideas could be used in many directions. So, as a pause before new developments, I would like to explain the genesis of this subject, at least from my point of view.

2. From (+) I recall that a figuration consists of

(1) A category $\underline{S}$ (of supports and deformations)

(2) A category $\underline{F}$ (of figures and substitutions)

(3) A functor $D: \underline{F}^{op} \times \underline{S} \longrightarrow SET$ (an element of

$D(F,S)$ is called a drawing of F in S)

(4) A functor $L: \underline{F} \longrightarrow \underline{C}$ injective and bijective on objects (an element of $\underline{C} \setminus L(\underline{F})$ is called a composition law).

If $(\underline{S},\underline{F},D,L,\underline{C}) = T$ is a figuration, a figurative algebra of type T or a T-algebra is a datum (S,A) where $S \in \underline{S}_o$, $A: \underline{C}^{op} \longrightarrow SET$ and

$$\begin{tikzcd} F^{op} \arrow[r, "L^{op}"] \arrow[dr, "D(-{,}S)"'] & \underline{C}^{op} \arrow[d, "A"] \arrow[dl, phantom, "="] \\ & SET \end{tikzcd}$$

So the T-algebra (S,A) is characterized by the datum for each composition law $c: F' \longrightarrow F \in \underline{C}_o$ of the action of c on the drawings:

$$A(c)_S : D(F,S) \longrightarrow D(F',S)$$
$$d \longmapsto dc \quad .$$

In the work toward this conception, the decisive step was to think of arities (of laws) and underlying sets (where the laws act) as abstract spaces, called resp. figures and supports, and to put the figures out of the category of supports; then for each support S the possible local domain of action of a law $c: F' \longrightarrow F$ has to be given a priori as a set $D(F,S)$.

A paradox in a figuration T is a figure $F \in \underline{F}_o$ such that the functor $D(F,-): \underline{S} \longrightarrow SET$ is not representable. If there are no paradoxes we are exactly in the case of algebraic structures over $\underline{S}$ (i. e. algebras of a monad on $\underline{S}$). But if we admit the existence of paradoxes in the figuration we get the possibility of describing in this operational way all first order theories over $\underline{S}$, and more: in fact the figurations over $\underline{S}$ describe the same things that possibly large $\underline{S}$-sketches $\underline{S} \longrightarrow //\underline{S}'//$. Grosso modo in the sketch $//\underline{S}'//$ associated to a figuration T the projective cones describe contacts between figures and the inductive cones describe potential motions of figures. Figurative algebras and sketches are resp. the synthetic and the analytic approach to the same "geometrical model theory" based on calculus of contacts, incidence relations and motions.

From a more philosophical point of view a figuration is a realization (by the drawings) of a dialogue between "ideal figures" and "real supports" (cf. the myth of the cavern of Plato). And because of that I feel that a lot of phenomenons in the nature consisting in antagonisms or dialectics can be formalized directly by figurations (this will be detailled in "Introduction à l'Analyse Algébrique I, II et III", with the help of a generalization of the calculus of Satellite Functors in the place of the classical differential calculus).

On one side, by the calculus of sketches we have a precise tool to control what can be hoped in a given figuration, and on the other side, this presentation by figuration seems to be too much general, and it is difficult to find an example of a theory which is not presentable by a figuration. Starting with a figuration T, and functors $E: \underline{F} \longrightarrow \underline{F}$, $R: \underline{S} \longrightarrow \underline{S}$, $W: SET \longrightarrow SET$, we get a new figuration $(\underline{S},\underline{F},\overline{D},L,\underline{C}) = \overline{T}$ by a prolongation of D in $\overline{D}$ defined by

$$\overline{D}(F,S) = W(D(E(F),R(S))).$$

In this way we get as examples of figurative algebras the vari-algebras (in SET), partial algebras, relational algebras, fuzzy algebras.

Other examples are the algebras in a monoïdal category, the categories (over graphs), the n-ary categories (over n-ary graphs), monoïdal categories (over categories), 1-connected spaces (over topological spaces), various elementary geometries, calculus of roots of equations, infinitesimal calculus (à la Elie Cartan) ...

In fact the T-algebras are the "very small" algebras of a monad on $SET^{(\underline{S}^*)}$ with $\underline{S}^* = (SET^{\underline{S}})^{op}$, and the study of figurations can be cut in two questions:

(1) The theory of fibered monads or monads on $FIB(\underline{X})$

(2) The theory of the size of objects in a category $FIB(\underline{X})$.

3. Perhaps the new thing in figurative algebras are the geometric coloration (Figures, supports, drawings, contacts, motions) and the insistence on the duality "Figures/Supports" (given by an arbitrary functor $D: \underline{F}^{op} \times \underline{S} \longrightarrow SET$) and so a "non-fondationalistic" approach of the theories as dialectics. But in fact this is the produce of several influences that I want to express now.

3.1. At first of course the classical works on sheaves and sketches, by A. Grothendieck, C. Chevalley, C. Ehresmann, and on monads and theories (especially the work of F. Linton). Then, the idea of the presentation of the algebraic laws by "extra"-arrows added to $\underline{F}$ (= the arrows of $\underline{C} \setminus L(\underline{F})$) is taken from the calculus of pro-monads of M. Thiébaud (1971) and the calculus of D-algebras of L. Coppey (1972-73) (but for these authors the laws act on Homs, because they are in the algebraic case). L. Coppey works with multiplicative graphs, and this is a way to express the local character of the laws. In the same time (1972-73) I have also developped the theory of "ébauches" and machines as a model of locally algebraic structures (i. e. structures where the laws are partially active). The ideas on partial algebras as expressed in the works of P. Burmeister, A. Obtulowicz, H. Andreka, I. Nemeti, has also something to do there. The description for each partial law $c: F' \longrightarrow F$ of its domain in the explicit form

$$\coprod_{S \in \underline{S}_o} D(F,S)$$

with the data $D(F,S)$ given abstractly a priori is new. But the visualization of the drawings as arrows $d: F \longrightarrow S$ and the construction of $\underline{S}' = \underline{F} \perp\!\!\perp_{D} \underline{S}$ the "joint" of D (used for the description of the sketch $//\underline{S}'//$ associated to T – see (+)) is taken from a lecture of C. Ehresmann in 1967-68 where he constructs the joint of two functors in order to express the notion of adjunction.

3.2. A second wave of influences started with the reading of works in Pattern Recognition (around 1973-74) of T. Pavlidis (on juxtaposition relations) and of A. Shaw (on Picture graphs and grammars) and continued by the reading of papers on Tree grammars and of the work of H. Ehrig, M. Pfender and H. Schneider on Graph-grammars (1973). In these times I worked also with a very particular object called GOS-formula or "formula over an oriented graph with a successor operator". In fact the first systematic logical study of theories over graphs is the work of G. Blanc between 1973 and 1975, followed by the diagrammatic languages of P. Freyd (1976). With the development of 2-theories by J. Gray, M. Kelly, A. & C. Ehresmann, this leads to try to work over graphs or categories in the place of sets. Of course the idea of working over an abstract category $\underline{S}$ is known (= theory of monads on $\underline{S}$, or theory of realizations of sketches in $\underline{S}$), but these influences clarify the concrete interest of this idea in the case $\underline{S}$ = GRAPH or $\underline{S}$ = CAT .

In 1979 for the case $\underline{S}$ = CAT a logical treatment was furnished by J. Penon, and a more algebraïc tool has been constructed by M. Kelly. In september 1979 I was in Czecho-Slovakia, and I learnt from V. Topentcharov what he was doing with n-ary categories; I realized that in fact he was working with special cases of what I called "simplicial algebras" (i. e. $\underline{S} = SET^{\Delta}$). This was for me the crucial meeting with algebras where the laws are submitted to some contact conditions (a previous meeting was in 1976 in my work with Topogenesis when I thought to put in parallel with the classical "words + equations" the sentence "figures + incidence relations").

In 1979-1981 A. Burroni worked on the case $\underline{S}$ = GRAPH and proved that the tripleability criterion of C. Lair works over GRAPH in several cases. At this time my problem was to understand if yes or no the case $\underline{S}$ = GRAPH (or, following the idea of M. Lazard on primary structures, the case SET^{Ω}) is fundamental. I was pushed in this conviction by the existence of the theory of spaces of R. Walters based on the description of abstract binary contacts of objects. But when I meet the work of R. Brown and F. Higgins on ω-groupoïds, the algebra of cubes and Van Kampen theorem (1981), I decided to come back seriously to the case $\underline{S} = SET^{\triangle}$ or $\underline{S} = SET^{\square}$; but, at first I had to inspect the general phenomenons for an arbitrary $\underline{S}$.

Fortunately for an arbitrary $\underline{S}$ something was done at this time by C. Lair and myself: the calculus of $\underline{S}$-formulas. But it was in an analytic shape. And finally I had to transposed the idea of $\underline{S}$-formula in a more synthetic framework. This have been done under the influence of readings in System Theory (M. Jessel, J. Eugène, L. v. Bertalanffy, E. Laszlo).

The last influence in the presentation of figurations is the fact that since around 3 years I work with L. van den Bril on the use of exact squares in order to understand the classical calculus of satellites. This is achieved now and - as I said in the §2 - we have obtained a tool well adapted to the study of figurations. But this is the beginning of another story.

4. Hereover I have explained my own experience with one idea - the idea of figuration - in order to strenghten by an example the opinion that generally ideas are the fruits of a large sheaf of influences. But it is also a fact that it is easy a posteriori to erase the objective influences living back of an idea, by a transformation of the formalism.
Let me show you how to do that with the idea of figuration.
In fact the datum $L: \underline{F} \longrightarrow \underline{C}$ is the same that the bifunctor $Hom_{\underline{C}}(L(-),L(-))$ denoted by $L': \underline{F}^{op} x \underline{F} \longrightarrow SET$,

and we have two bimodules $D: \underline{F} \longrightarrow\!\!\!\!\!\!\!+\;\; \underline{S}$ and $L': \underline{F} \longrightarrow\!\!\!\!\!\!\!+\;\; \underline{F}$ and their composite $D \boxtimes L' = P$. In order to define the algebras we need D, P, the inclusion $D \hookrightarrow P$, and it is not needed to know that P is decomposable as $D \boxtimes L$. So the part played by the laws (given by L) can be eliminated, and the definition of §2 can be reformulated as follows:

Definition. A figuration is the data of
$$\begin{tikzcd} \underline{F} \arrow[r, bend left, "D"{name=A}] \arrow[r, bend right, "P"'{name=B}] & \underline{S} \arrow[from=A, to=B, Rightarrow, "u"] \end{tikzcd}$$
where $\underline{F}$ is the category of figures, $\underline{S}$ is the category of supports, D is the bimodule of drawings, P is the bimodule of potential drawings and $u: D \longrightarrow P$ says that drawings are potential drawings. Then an algebra of this figuration is a (S,A) with $S \in \underline{S}_o$ and A a co-section of $\ulcorner S \urcorner \circ \boxtimes u : \ulcorner S \urcorner \circ \boxtimes D \longrightarrow \ulcorner S \urcorner \circ \boxtimes P$ (where $\ulcorner S \urcorner : \mathbb{I} \longrightarrow \underline{S}$ and $\ulcorner S \urcorner \dashv \ulcorner S \urcorner \circ$ as bimodules).

Let us recall that if
$$\begin{tikzcd} X \arrow[rr, "G"] \arrow[dr, "P"'] & \arrow[d, phantom, "\overset{g}{\Rightarrow}"] & Y \arrow[dl, "Q"] \\ & (SET^T)^{op} & \end{tikzcd}$$
is a T-formula of a concrete sketch then $F: T \longrightarrow SET$ is a model of g iff $(\ulcorner F \urcorner \circ) \boxtimes \tilde{g}$ is an iso:
$$\begin{tikzcd} & X \arrow[dr, "P"] \arrow[d, phantom, "g\Downarrow"] & & \\ Y \arrow[ur, "G\circ"] \arrow[rr, "Q"'] & {} & (SET^T)^{op} \arrow[r, "\ulcorner F \urcorner \circ"] & \mathbb{I} \end{tikzcd}$$

So in this presentation the existence of a structure (here an algebra) extends the notion of satisfaction of a formula (i. e. model of a T-formula).
