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CONSTRUCTION OF AN HOMOLOGY AND A COHOMOLOGY THEORY ASSOCIATED TO A FIRST ORDER FORMULA

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RESUME - On montre comment chaque formule ϕ d'un langage $\mathcal{L}$ détermine une théorie d'homologie (et une théorie de cohomologie) sur la catégorie des interprétations de $\mathcal{L}$, dont la valeur sur chaque interprétation I de $\mathcal{L}$ est une obstruction à $I \models \phi$ "à des co-équations près" (et "à des équations près").

This paper is a sequel of [7].

0. Let $\mathcal{L}$ be a first order language, let I be an interpretation of $\mathcal{L}$, and let ϕ be a formula of $\mathcal{L}$. The aim of this note is to indicate a way in which it is possible to measure partially and to compute how I is far from the models of ϕ.

In the papers [7] and [8] it is shown how this question is connected with the possibility of a geometrical study of algorithms and ambiguïties.

1. PROPOSITION. *Let* $\mu(x_1,\ldots,x_n)$ *be a first order formula of* $\mathcal{L}$, *and let* $\mathrm{Mod}_\mu\phi$ *be the category with objects the models of* ϕ, *and with morphisms from* M *to* M' *the morphisms (of models of* Φ*)* $m : M \longrightarrow M'$ *such that*

$$\forall\, x_1,\ldots,x_n\ [\ \mu(m(x_1),\ldots,m(x_n)) \longrightarrow \mu(x_1,\ldots,x_n)\ ]$$

Then there is a small mixed sketch σ *such that* $\mathrm{Mod}_\mu\phi \cong \mathrm{Mod}\sigma$.

The existence of σ is proved by the juxtaposition of proposition 3 p.8 of [6], théorème 2.1 p.26 of [5], and proposition 3 p.301 of [7],II. In fact, this juxtaposition shows more than our proposition here.

2. For **C** a category, let BC = |NC| be the geometric realization of the nerve of **C**. BC is a cw-complexe, and $\pi_1 BC \cong C[C^{-1}]$ (the category of fractions of **C**). Of course if **C** is a class, BC is a class too. But, if **C** = Modσ for a small sketch σ, then in BC we can construct a *set* gσ such that the inclusion gσ $\longrightarrow$ BModσ is an equivalence of homotopy. In particular we get

PROPOSITION. Modσ[(Modσ)$^{-1}$] *is a small groupoïd, up to equivalence. We call it the fundamental groupoïd of* σ, *and we denote it by* $\pi_1 g\sigma$.

The existence of the set gσ comes from [7].

3. Let $Mod_\mu \phi / I$ be the category with objects the morphisms (of interpretations of $\mathcal{L}$) f : M $\longrightarrow$ I where M is a model of ϕ, and with morphisms, from f : M $\longrightarrow$ I to f' : M' $\longrightarrow$ I, the morphisms of models (morphisms of $Mod_\mu \phi$) g : M $\longrightarrow$ M' such that f'.g = f.
Then

PROPOSITION. *There is a small sketch* $\sigma = \sigma(\mathcal{L}, I, \phi, \mu)$ *such that*

$$Mod_\mu \phi / I \cong Mod\sigma.$$

So we get a small cw-complexe $g\sigma(\mathcal{L}, I, \phi, \mu)$, *which is a geometric description of the position of* I *with respect to* $Mod_\mu \phi$.

4. Let **Ab** be the category of small abelian groups, and let F : $Mod_\mu \phi$ $\longrightarrow$ **Ab** be a functor. (In particular F could be the constant functor on a fixed abelian group A, or it could be a "canonical" functor if $\mathcal{L}$ is a language over the language of abelian groups, etc).
The André's homology measures "how I is far from $Mod_\mu \phi$, from the point of view of F".In order to do that we consider the chain complex

$$\longrightarrow C_2(I,F) \xrightarrow{d_2} C_1(I,F) \xrightarrow{d_1} C_0(I,F) \xrightarrow{d_0} 0$$

which is

$$\ldots \longrightarrow \sum_{M_2 \to M_1 \to M_0 \to I} FM_2 \xrightarrow{d_2} \sum_{M_1 \to M_0 \to I} FM_1 \xrightarrow{d_1} \sum_{M_0 \to I} FM_0 \xrightarrow{d_0} 0$$

with $d_1 = s_1^0 - s_1^1$, where

$$s_1^0 : (FM_1)(M_1 \xrightarrow{\alpha} M_0 \xrightarrow{\beta} I) \xrightarrow{Id} (FM_1)(M_1 \xrightarrow{\beta.\alpha} I) \xrightarrow{inc} \sum_{M \to I} FM$$

$$s_1^1 : (FM_1)(M_1 \xrightarrow{\alpha} M_0 \xrightarrow{\beta} I) \xrightarrow{F(\alpha)} (FM_0)(M_0 \xrightarrow{\beta} I) \xrightarrow{inc} \sum_{M \to I} FM$$

and so on, and we define

$H_0(I, F) = \ker d_0/\mathrm{Im}\, d_1 = \mathrm{coker}\, d_1$, $H_1(I, F) = \ker d_1/\mathrm{Im}\, d_2$, and, for every $n \geq 0$, $H_n(I, F) = \ker d_n/\mathrm{Im}\, d_{n+1}$.

PROPOSITION. $H_n(I, F)$ *is a function of* F, I, μ, ϕ, *which in fact depends only of the homotopy type of* $Mod_\mu\phi/I$ *and of* F *and could be denoted by* $H_n(Mod_\mu\phi/I,F)$.

see [1], [2], [3] and [4].

Let $Int\mathscr{L}$ be the category of interpretations of $\mathscr{L}$, let $J : Mod\phi \longrightarrow Int\mathscr{L}$ be the canonical inclusion. Then the inductive Kan extension of F along J is given by

$$[\underrightarrow{Ext}_J F](I) = \operatorname*{Lim}_{M_0 \longrightarrow I} F(M_0)$$

and we have

$$H_0(I, F) = [\underrightarrow{Ext}_J F](I).$$

If $I \models \phi$, then $H_n(Mod_\mu\phi/I,F) = \begin{cases} F(I) \text{ if } n = 0 \\ 0 \quad \text{if } n > 0 \end{cases}$

5. Now, the point is that, because of the results hereover (§§ 1 to 4), we get

PROPOSITION. *The tools of* [1] *and of* [3], available in the situation where a <u>full</u> <u>and</u> <u>small</u> category **M** (called a category of "models") lives inside a big category of "spaces", *are also available in the situation where a* (<u>possibly</u> <u>big</u> <u>and</u> <u>not</u> <u>necessarly</u> <u>full</u>) *category* $\mathrm{Mod}_\mu\phi$ *of models of a theory lives inside a big category of interpretations of a language* $\mathcal{L}$ (compare with the idea of "paires adéquates" p. 43 of [1]). *Precisely here we get the fact that the* $H_n(\mathrm{Mod}_\mu\phi/I,F)$ *are small.*

6. After the existence of $g\sigma$ proved in [7], the theorem hereunder §9 is just a second stone for a work to be pursed. Theoretically the computation of our H_n is based on the effective construction of a "locally cofree diagram", and more precisely on the construction of a "relatively cofiltered locally cofree diagram" (r.cf.l.cf.d.)(see [5] and [6]) (in the category $\mathrm{Mod}_\mu\phi$) generated by I. This r.cf.l.cf.d. contains all the information we need, and it will be the starting point of an absolute calculus. But for concrete situations we need a relative calculus, by the way of comparaisons between various H_n. For that it will be essential to go toward effective relative calculation of these small H_n, and especially we need a description of the link between these calculations and the theory of demonstrations. For example we need relations among $H_n(\mathrm{Mod}_\mu\phi/I,F)$, $H_n(\mathrm{Mod}_\mu\gamma/I,G)$, $H_n(\mathrm{Mod}_\mu[\phi\wedge\gamma]/I,K)$, $H_n(\mathrm{Mod}_\mu[\phi\Rightarrow\gamma]/I,L)$ (for conveniant K and L).
For that it will be necessary to describe the category **For**($\mathcal{L}$) of formulas of the language $\mathcal{L}$. At first this will be usefull to precise the functoriality of the $H_n(\mathrm{Mod}_\mu\phi/I,F)$ with respect to ϕ and μ.

7.The first purpose of this paper was to show precisely how each classical first order formula ϕ of a language $\mathcal{L}$ determines a "small" homology theory on the category of interpretations of $\mathcal{L}$.
Now, the continuation of this research pass trough the description of **For**($\mathcal{L}$). With respect to that, I would like to make the following remark : what have to be morphisms between formulas ? it is not so clear a priori ; they have to be "demonstrations" or "proofs", but there is no

canonical idea of what is a demonstration.

But if we decide to stay in (or to come back to) the style of sketches, a first picture is easy to give. In fact $\mathcal{L}$ "is" a sketch σ_0 (i.e. the category of interpretations of $\mathcal{L}$ is isomorphic to Modσ_0), the formula ϕ (or ${}_\mu\phi$) is a sketch σ, and the inclusion of the category of models of ϕ (of ${}_\mu\phi$) in the category of interpretations of $\mathcal{L}$ is induced by a morphism of sketches P : $\sigma_0 \longrightarrow \sigma$. This P is the "proof" that a model of ϕ (of ${}_\mu\phi$) is an interpretation of $\mathcal{L}$. In fact P is not a general morphism of sketches, but determines σ as a σ_0-sketch (see [6] p.10 for the precise definition). So we choose to say now that a formula for σ_0 (in the place of a $\mathcal{L}$-formula) is nothing but such a P, a σ_0-sketch. In [6] the boolean calculus of σ_0-sketches (conjonctions, disjonctions, complements) is exposed as construction in the category of sketches. Then we can defined the category **For**(σ_0) as being the category of σ_0-sketches, as objects, with morphisms from P to P' the morphisms of sketches f : $\sigma \longrightarrow \sigma'$ which determine σ' as a σ-sketch, such that f.P = P'.

At this level of language, we can change our notations, replacing Mod$_\mu\phi$ by Modσ, or even, more precisely, by P, and the H_n(Mod$_\mu\phi$/I,F) will be denoted by H_n(P/I,F). Of course for general mixed sketches (and not only for those associated to first order formulas) the result in §5 works, and the abelian groups H_n(P/I,F) are small. Now

PROPOSITION. *The functoriality of these* H_n, *with respect to* P, I *and* F *are trivial facts.*

8. In a dual way, given a functor F : Mod$_\mu\phi$ $\longrightarrow$ **Ab** and an interpretation I of $\mathcal{L}$, the cohomology of I with coefficient in F is defined by considering the cochain complex

$$\longleftarrow C^2(I,F) \xleftarrow{d^1} C^1(I,F) \xleftarrow{d^0} C^0(I,F)$$

which is

$$\longleftarrow \prod_{M_2 \leftarrow M_1 \leftarrow M_0 \leftarrow I} FM_2 \overset{d^1}{\longleftarrow} \prod_{M_1 \leftarrow M_0 \leftarrow I} FM_1 \overset{d^0}{\longleftarrow} \prod_{M_0 \leftarrow I} FM_0$$

with

$$d^1(x)_{(I \xrightarrow{\chi} M_0 \xrightarrow{\lambda} M_1)} = F(\lambda)(x_{(I \xrightarrow{\chi} M_0)}) - x_{(I \xrightarrow{\lambda.\chi} M_1)}$$

and so on, and we define $H^n(I, F) = \ker d^n / \mathrm{Im}\, d^{n-1}$.

For these cohomology groups, the same result is true, that is to say that they are small. But now, the computation is based on the effective construction of a "relatively filtered locally free diagram" (r.f.l.f.d.) (in the category $\mathrm{Mod}_\mu\phi$) generated by I. These cohomolgy groups will be denoted by $H^n((I/\mathrm{Mod}_\mu\phi)^{op},F)$.

9.Collecting the results of §5, §7 and §8, we get :

THEOREM :*The abelian groups* $H_n(\mathrm{Mod}_\mu\phi/I,F)$ *and* $H^n((I/\mathrm{Mod}_\mu\phi)^{op},F)$ *are small, i.e. they are elements of the category* **Ab**, *they are functorial with respect to* I, F, μ *and* ϕ, *and if* $I \models \phi$, *then* $H_n(\mathrm{Mod}_\mu\phi/I,F) = 0$, *for every* $n > 0$, *and* $H^n((I/\mathrm{Mod}_\mu\phi)^{op},F) = 0$, *for* $n > 0$. *In fact, more precisely, we have* $H_n(\mathrm{Mod}_\mu\phi/I,F) = 0$, *for every* $n > 0$, *if there is a cofree model generated by* I, *and we have* $H^n((I/\mathrm{Mod}_\mu\phi)^{op},F) = 0$, *for* $n > 0$, *if there is a free model generated by* I. *So they are small obstructions to the satisfaction of* ϕ *in* I *"up to co-equations" and "up to "equations".*

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